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Time-Varying Risk, Interest Rates and Exchange Rates in General Equilibrium*

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ABSTRACT

We argue that the data on interest rates and exchange rates implies that time-varying risk is the primary force driving nominal interest differentials on different currency denominated bonds. Since exchange rates are roughly random walks, the risk premium on a foreign currency denominated bond is roughly the interest differential over a home currency denominated bond plus a constant. Thus variations in the interest differential are driven almost entirely by variations in the risk premium. Moreover, the tendency for higher interest rate currencies to appreciate is difficult to explain without large and systematic variations in risk. To address these issues we build a general equilibrium model with fixed costs to exchanging money for assets. As the underlying shocks to money growth vary so does the number of agents who participate in the asset market. This variation leads to risk in the economy to vary systematically with the level of the inflation rate. We show that if this variation is sufficiently large then the model can produce the key time-series features of interest rates and exchange rates. We show that some cross section evidence supports the key mechanism at work in the model.

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"Overall, the new view of finance amounts to a profound change. We have to get used to the fact that most returns and price variation comes from variation in *risk premia...*" (Cochrane, p.451)

We develop a monetary general equilibrium model which generates time varying risk premia as a result of endogenous market segmentation. That time-varying risk is essential for understanding the movements in asset prices has been widely documented. To illustrate the basic workings of the model we apply it to interest rates and exchange rates since data on these variables provide some of the most compelling evidence for the importance of time-varying risk premia.

To see why time-varying risk premia is important consider the evidence that comes from nominal interest rates and exchange rates. To be concrete consider the risks faced by an investor choosing between investing in dollar denominated bonds and euro denominated bonds. Clearly, the dollar return on the euro bond is risky because next period's exchange rate is not known today. The risk premium compensates the holder of this bond for this exchange rate risk and is defined as the expected log dollar return on a euro bond minus the log dollar return on a dollar bond.[1] In logs, this *risk premium* is

$$p_t = i_t^* + E_t \log e_{t+1} - \log e_t - i_t.$$

where i_t^* and i_t are the euro and dollar interest rates and e_t is the exchange rate. It follows from the definition of the risk premium that the difference in nominal interest rates across currencies can be divided into the expected change in the exchange rate between these currencies and a currency risk premium.

In standard equilibrium models of interest rates and exchange rates, risk premia are constant, hence movements in interest rate differentials move one-for-one with the expected change in the exchange rate. The data suggests nearly the opposite. In the data exchange rates are roughly a random walk in that the expected depreciation of a currency, $E_t \log e_{t+1} - \log e_t$, is roughly constant[2]. Thus, the interest rate differential $i_t^* - i_t$ is approximately the risk premium p_t plus a constant. Hence, the observed variations in the interest differential are *almost entirely* accounted for by movements in the risk premium.

The idea that currency risk premia must be highly variable originally stems from the observation that high interest rate currencies tend to appreciate. This observation, documented by Fama (1984) and Hodrick (1987) among others, is widely referred to as the *forward premium anomaly.* Without large variations in risk, this tendency cannot be explained since investors would demand higher interest rates on currencies that are expected to fall in value, not ones that are expected to rise in value.

The basic idea that asset market are segmented in the sense that, at any given time, only a fraction of agents participate in them, has been used in a variety of settings to account for the high levels of risk premia. (See ......) Here we develop a general equilibrium monetary model that generates time-varying risk premia as a result of variations in the degree of market segmentation that arise endogenously in response to changes in the money growth rate. We show that the model can generate, qualitatively, the type of systematic variations in risk premium called for by the data on interest rates and exchange rates. Rather than to build a quantitative model, we deliberately build a simple model where the main mechanism can be clearly seen. For example, we abstract from trade in goods throughout so as to focus on frictions in asset markets.

Our model is a two country pure exchange cash-in-advance economy. The key difference between this model and the standard cash-in-advance model is that agents must pay a fixed cost to transfer money between the goods market and the asset market. This fixed cost is similar to that in the models of Baumol (1952) and Tobin (1956). This fixed cost differs across agents. In each period agents with fixed costs below some cutoff level pay the fixed cost and thus, at the margin, freely exchange bonds and money. Agents with fixed costs higher than the cutoff level choose not to pay the fixed cost and hence do not. In this sense, asset markets are segmented.

Increases in the money growth rate increase the inflation rate which in our model increases the net benefit of participating in the asset market. Through this mechanism an increase in money growth leads more agents to participate actively in asset markets and hence lowers the risk premium. We show that this type of time-varying risk can be the primary force driving interest rate differentials. Moreover, this time-varying risk also generates the forward premium anomaly.

So far we have discussed our model's implications for the time series variability for the risk premium. The mechanism for this variability is that at times when money growth and inflation are relatively high the fraction of agents that are participating in the asset market is relatively high and hence risk is relatively low. Our model also have implications for the cross section. One implication of our model's key mechanism is that if inflation is permanently higher in one country then asset market participation is also permanently higher. With higher asset market participation the marginal utility of active agents is less sensitive to money growth fluctuations and thus the volatility of the risk premium should be small. Thus, the model predicts that countries with high enough inflation should not have a forward premium anomaly. In some work comparing the forward premium in developed and emerging economies, Bansal and Dahlquist (2000) find evidence supporting this prediction.

In terms of the literature there are a number of monetary equilibrium models with (roughly) constant risk premia. As Backus, Foresi and Telmer (1995) and Engel (1996) emphasize, these models cannot address the features of the date discussed here. The contribution of this paper is to develop such a model. We have shown that by taking the standard cash-in-advance model and adding one key ingredient, the endogenous market segmentation arising from the fixed cost, we transform a model that is inconsistent with the most basic features of the data to one that is at least potentially consistent with many of them.

One way to generate time-varying risk is to have the conditional variances of the underlying shocks vary greatly with their levels (See Obstfeld and Rogoff 1998.) As Hodrick (1989) and Backus, Foresi and Telmer (1995) have noted that there is little evidence in the data for such movements of the appropriate magnitude. Motivated by this evidence we take a different approach. In our model the risk premium is time-varying even though the underlying shocks have constant conditional variances.

Our work builds on that of Rotemberg (1985), Alvarez and Atkeson (1997) and it most closely related to that in Alvarez, Atkeson and Kehoe (2002). It is also related to the work of Grilli and Roubini (1992) and Schlagenhauf and Wrase (1995) who study the effects of money injections on exchange rates in two country variants of the models in Lucas (1990) and Fuerst (1992) but do not address variations in the risk premium.

1. Risk, Interest Rates and Exchange Rates in the Data

We have argued that interest differentials are driven mainly by time-varying risk. To make our argument precise we first define the (log) risk premium for a euro denominated bond as the expected log dollar return on a euro bond minus the log dollar return on a dollar bond. The dollar return on a euro bond, $(1+i_t^*)e_{t+1}/e_t$, is obtained by converting a dollar at t to $1/e_t$ euros, buying a euro bond paying interest $1+i_t^*$ and then converting the resulting euros back to dollars at $t+1$ at exchange rate e_{t+1}. Hence, in logs, the *risk premium* is

$$(1) \qquad p_t = i_t^* + E_t \log e_{t+1} - \log e_t - i_t.$$

Clearly, the dollar return on the euro bond is risky because the future exchange rate e_{t+1} is not known at t. The risk premium compensates the holder of this bond for this exchange rate risk.

Our argument follows from combining the definition of the risk premium with two features of the data, namely that exchange rates are close to martingales, in the sense that the variance of $E_t \log e_{t+1} - \log e_t$ is small, and the variance of interest differentials is fairly large. To demonstrate our claim first consider the extreme case in which the exchange rate is exactly a martingale so that $E_t \log e_{t+1} - \log e_t$ is constant. Then (5) implies that $i_t^* - i_t$ is just the risk premium plus a constant, so that literally all of the movements in the interest differential is driven by variations in the risk premium. In the more relevant case in which variance of expected depreciation is small, we see that most of the movements in the interest differential are driven by movements in the risk premium.

A related feature of the data is that high interest rate currencies tend to appreciate in that

$$(2) \qquad cov\left(i_t - i_t^*, \log e_{t+1} - \log e_t\right) \leq 0$$

where i_t and i_t^* are the nominal interest rates on home and foreign currency and e_t is the price of foreign currency in units of domestic currency. This feature has been widely documented for the currencies of the major industrialized countries over the period of floating exchange rates. (See, for example, Backus, Foresi, and Telmer 1998 for a recent discussion). The inequality (2) is referred to as the *forward premium anomaly.* [3] Notice that (2) is equivalent to

$$(3) \qquad cov\,(i_t - i_t^*, E_t \log e_{t+1} - \log e_t) \leq 0$$

which is more convenient to work with in a theoretical model. In the literature this anomaly is documented by a regression of the change in the exchange rates on the interest differential

$$(4) \qquad \log e_{t+1} - \log e_t = a + b(i_t - i_t^*) + v_{t+1}.$$

Such regressions typically yield estimates of b that are zero or negative. (These regressions also have low R^2 and so in this sense the forward premium anomaly is consistent with the fact that exchange rates are close to martingales.)

Notice that if the risk premium is constant, then movements in the expected depreciation and movements in the interest differential are perfectly correlated. In particular, if p_t is constant than if $E_t \log e_{t+1} - \log e_t$ falls then $i_t - i_t^*$ also falls and the covariance in (3) is positive. Clearly, a successful theory of the forward premium anomaly must have that time-varying risk premia. In particular, when there is an increase in the expected appreciation of the dollar, the risk premium must fall so much that dollar interest rates rise relative to euro interest rates. Mechanically, for $E_t \log e_{t+1} - \log e_t$ to be negatively correlated with

$$(5) \qquad i_t - i_t^* = E_t \log e_{t+1} - \log e_t - p_t$$

it must be that as $E_t \log e_{t+1} - \log e_t$ falls the risk premium p_t falls by so much that $i_t - i_t^*$ rises.

To develop the connection between expected depreciation and risk premia more fully we find it useful to use (1) to rewrite (3) as

$$(6) \qquad cov(E_t \log e_{t+1} - \log e_t, p_t) \geq var(E_t \log e_{t+1} - \log e_t).$$

To get some intuition about what (6) entails notice that it implies the following two conditions: i) when the home currency appreciates the risk premium falls (so that the covariance in (6) is positive) and ii) movements in the risk premium are large in that

$$(7) \qquad var\,(p_t) \geq var\,(i_t - i_t^*)\,.$$

To derive (7) substitute (1) into (6), manipulate the resulting inequality to be

$$var\,(i_t - i_t) \leq cov\,(i_t - i_t^*, p_t) = corr\,(i_t - i_t^*, p_t)\,std\,(i_t - i_t^*)\,std\,(p_t)\,,$$

then divide by $std\,(i_t - i_t^*)$ use the fact that a correlation is less than or equal to one.

2. The economy

Consider a two country, cash-in-advance economy with an infinite number of periods $t = 0, 1, 2, \ldots$. We refer to one country as the home country and the other as the foreign country. In each country, there is a government and a continuum of households of measure one. Households in the home country use the home currency, which we refer to as dollars, to purchase a home good. Households in the foreign country use the foreign currency, which we refer to as euros, to purchase a foreign good.

Trade in this economy at dates $t \geq 1$ takes place in three separate locations: an asset market and the two goods markets. In the asset market, households trade the two currencies and dollar and euro bonds which promise delivery of the relevant currency in the asset market in the next period, and the two governments introduce their currencies via open market operations. In each goods market, households use the local currency to buy the local good subject to a cash-in-advance constraint and sell their endowment of the local good for local currency.

Each household must pay a real fixed cost γ for each transfer of cash between the asset market and the goods market. This fixed cost is constant over time for any specific household but varies across households in both countries according to a distribution with density $f(\gamma)$. Households are indexed by their fixed cost γ. The fixed costs for households in each country are in units of the local good.

The only source of uncertainty in this economy is the money growth shocks in the two countries. The timing within each period $t \geq 1$ for a household in the home country is illustrated in Figure 1. We emphasize the physical separation between markets by placing the asset market in the top half of the picture and the goods market in the bottom half. Households in the home country enter the period with the cash $P_{-1}y$ they obtained from selling their endowments at $t-1$, where P_{-1} is the price level and y is their endowment. Each government conducts an open market operation in the asset market which determines the realizations of money growth μ and μ^* in the two countries and the current price levels in the two countries P and P^*.

The household then splits into a worker and a shopper. The worker sells the household endowment y for cash Py and rejoins the shopper at the end of the period. The shopper takes the household's cash $P_{-1}y$ with real value $n = P_{-1}y/P$ and shops for goods. The shopper can choose to pay the fixed cost γ to transfer cash Px with real value x to or from the asset market. This fixed cost is paid in cash obtained in the asset market. If the shopper pays the fixed cost then the cash in advance constraint is $c = n + x$; otherwise this constraint is $c = n$.

The household also enters the period with bonds that are claims to cash in the asset markets with payoffs contingent on the rates of money growth μ and μ^* in the current period. This cash can either be reinvested in the asset market or, if the fixed cost is paid, can be transferred to the goods market. In Figure 1, letting B denote the current realization of the state-contingent bonds and $\int qB'$ the household's purchases of new bonds, the asset market constraint is $B = \int qB' + P[x + \gamma]$ if the fixed cost is paid and $B = \int qB'$ otherwise. At the beginning of period $t + 1$, this household starts with cash Py in the goods markets and contingent bonds B' in the asset market.

In equilibrium households with sufficiently low fixed costs pay this cost and transfer cash between the goods and asset markets while others do not. We refer to households that pay the fixed cost as *active* and refer to households who do not as *inactive*. Inactive households simply consume their current real balances.

In the rest of the section we flesh out this outline of the economy. This model is related to that in Alvarez, Atkeson and Kehoe (2002). In that model all agents face the same fixed cost but i.i.d. idiosyncratic income shocks and households choose to be active or inactive based on the realizations of these shocks.

Throughout the paper we assume that the shopper's cash-in-advance constraint binds and that in the asset markets households hold their assets in interest-bearing securities rather than cash. Alvarez, Atkeson and Kehoe (2002) provide sufficient conditions to this to be true in that related model. It is easy to extend those arguments to this model. One might also consider variants of this model in which the fixed cost for each household varies randomly over time. For the appropriate set of sufficient conditions the cash-in-advance constraints would always bind in those variants and the equilibrium would be identical.

At the beginning of period 1, home households of type γ have M_0 units of home money, $\bar{B}_h(\gamma)$ units of the home government debt and $\bar{B}_h^*$ units of the foreign government debt which are claims on $\bar{B}_h(\gamma)$ dollars and $\bar{B}_h^*$ euros in the asset market at that date. Likewise, foreign households start period 1 with M_0^* euro holdings in the foreign goods market and start period 0 with $\bar{B}_f$ units of the home government debt and $\bar{B}_f^*(\gamma)$ units of the foreign government debt in the asset market in period 0.

We let M_t denote the stock aggregate supply of dollars in period t, and let $\mu_t = M_t/M_{t-1}$ denote the growth rate of the dollar supply. Similarly, let μ_t^* be the growth rate of the supply of euros M_t^*. Let $s_t = (\mu_t, \mu_t^*)$ and let $s^t = (s_1, \ldots, s_t)$ denote the history of money growth shocks up through period t and let $g(s^t)$ denote the density of the probability distribution over such histories.

The home government issues one period dollar bonds contingent on the aggregate state s^t. At date t, given state s^t, the home government pays off outstanding bonds $B\left(s^t\right)$ in dollars and issues claims to dollars in the next asset market of the form $B(s^t, s_{t+1})$ at prices $q(s^t, s_{t+1})$. The home government budget constraint at s^t with $t \geq 1$ is

$$(8) \qquad B(s^t) = M\left(s^t\right) - M(s^{t-1}) + \int_{s_{t+1}} q(s^t, s_{t+1}) B\left(s^t, s_{t+1}\right) ds_{t+1},$$

with $M(s^0) = \bar{M}$ given and, at $t = 0$ the constraint is $\bar{B} = \int_{s_1} q(s^1) B(s^1) ds_1$. Likewise, the foreign government issues euro bonds denoted $B^*(s^t)$ with bond prices denoted $q^*(s^t, s_{t+1})$. The budget constraint for the foreign government is then parallel to the constraint above.

In the asset market at each date and state, home households trade a set of one period dollar bonds and euro bonds that have payoffs next period contingent on the aggregate event

s_{t+1}. Arbitrage between these bonds implies that

$$(9) \qquad q(s^t, s_{t+1}) = q^*(s^t, s_{t+1}) e(s^t)/e(s^{t+1})$$

and thus without loss of generality we can assume that home households trade in home bonds and foreign households trade in foreign bonds.

Consider now the problem of an household of type γ in the home country. Let $P(s^t)$ denote the price level in dollars the home goods market at date $t \geq 1$. In each period $t \geq 1$, in the goods market households of type γ start the period with dollar real balances $n(s^t, \gamma)$. They then choose transfers of real balances between the goods market and the asset market $x(s^t, \gamma)$, an indicator variable $z(s^t, \gamma)$ equal to zero if these transfers are zero and one if they are not and consumption of the home good $c(s^t, \gamma)$ subject to the cash-in-advance constraint

$$(10) \qquad c(s^t, \gamma) = n(s^t, \gamma) + x(s^t, \gamma) z(s^t, \gamma),$$

$$(11) \qquad n\left(s^{t+1}, \gamma\right) = \frac{P(s^t) y}{P(s^{t+1})}.$$

where in (10) at $t = 1$, the term $P(s^1) n(s^1, \gamma)$ is given by M_0. In the asset market at $t \geq 1$, home households begin with cash payments $B(s^t, \gamma)$ on their bonds. They purchase new bonds and make cash transfers to the goods market subject to the sequence of budget constraints

$$(12) \qquad B\left(s^t, \gamma\right) = \int_{s_{t+1}} q(s^t, s_{t+1}) B\left(s^t, s_{t+1}\right) ds_{t+1} + P(s^t) \left[x(s^t, \gamma) + \gamma\right] z(s^t, \gamma).$$

We assume that both consumption and real bondholdings $B(s^t, \gamma)/P(s^t)$ are uniformly bounded by some large constants.

The problem of a home consumer of type γ is to maximize

$$(13) \qquad \sum_{t=1}^{\infty} \beta^t \int U(c(s^t, \gamma)) g(s^t) d\mu^t$$

subject to the constraints (10)- (12). Consumers in the foreign country solve the analogous problem with $P^*(s^t)$ denoting the price level in the foreign country in euros. We require that $\int \bar{B}_h(\gamma) f(\gamma) d\gamma + \bar{B}_f = \bar{B}$ and $\bar{B}_h^* + \int \bar{B}_f^*(\gamma) f(\gamma) d\gamma = \bar{B}^*$.

Since each transfer of cash between the asset market and the home goods market consumes γ units of the home good, the total goods cost of carrying out all transfers between home consumers and the asset market at t is $\gamma \int z(s^t, \gamma) f(\gamma) d\gamma$, and likewise for the foreign consumers. The resource constraint in the home country is given by

$$(14) \quad \int \left[c(s^t, \gamma) + \gamma z(s^t, \gamma) \right] f(\gamma) d\gamma = Y$$

for all t, s^t, and we have the analogous constraint in the foreign country. The fixed costs are paid for with cash obtained in the asset market and thus the home country money market clearing condition at $t \geq 0$ is given by

$$(15) \quad \int \left(n(s^t, \gamma) + \left[x(s^t, \gamma) + \gamma \right] z(s^t, \gamma) \right) f(\gamma) d\gamma = M(s^t)/P(s^t)$$

for all s^t. The money market clearing conditions for the foreign country are analogous. We let c denote the sequences of functions $c(s^t, \gamma)$ and use similar notation for other variables.

An *equilibrium* is a collection of bond and goods prices q, q^* and P, P^* together with bondholdings B, B^* for individuals B, B^* for the government, and an allocation c, x, z, n and c^*, x^*, z^*, n^* such that for each γ, the bond holdings and the allocation solve the households' utility maximization problems, the governments' budget constraints hold, and the resource constraints and the money market clearing conditions are satisfied.

3. Characterizing equilibrium

Here we solve for the consumption and real balances of active and inactive households. We then characterize the link between the consumption of active households and asset prices.

Throughout we assume that the cash-in-advance constraint always bind and the households hold only interest-bearing securities in the asset market. (We can provide sufficient conditions for this to hold using an argument similar to that given in Alvarez, Atkeson and Kehoe 2002.) Under this assumption, a household's decision to pay the fixed cost to trade at date t affects only its current consumption and bondholdings and does not directly affect the real balances it holds in the goods market at later dates. Notice that the constraints (11), (14) and (15) imply that the price level is

$$(16) \quad P(s^t) = M(s^t)/Y,$$

the inflation rate is $\pi_t = \mu_t$, real money holdings are $n(s^t, \gamma) = y/\mu_t$. Hence the consumption of inactive households is $c(s^t, \gamma) = y_{t-1}/\mu_t$. Let $c_A(s^t, \gamma)$ denote the consumption of an active household for a given s^t and γ.

In this economy inflation is distorting because it reduces the consumption of any household that chooses to be inactive. This effect induces some households to use real resources to pay the fixed cost thereby reducing the total amount of resources available for consumption. There are no other distortions beyond this one. Because of this feature it turns out that the competitive equilibrium allocations and asset prices can be found from the solution of the following planning problem for the home country together with the analogous problem for the foreign country

$$\max \sum_{t=1}^{\infty} \beta^t \int_{s^t} \int U(c(s^t, \gamma) f(\gamma) g(s^t) d\gamma ds^t$$

subject to the resource constraint (14) and

$$(17) \quad c(s^t, \gamma) = z(s^t, \gamma) c_A(s^t, \gamma) + (1 - z(s^t, \gamma)) y/\mu_t.$$

The constraint (17) captures the restriction that the consumption of households that do not pay the fixed cost is pinned down by their real balances y/μ_t. Here we have the planning weight for households of type γ simply being the fraction of agents of this type.

This problem can be decentralized with the appropriate settings of the initial endowments $B(\gamma)$ and $B^*(\gamma)$. Asset prices are obtained from the multipliers on the resource constraints above. Formally, we can show that the solution to this planning problem is an equilibrium by directly verifying that it satisfies the equilibrium condition.

Notice that this problem reduces to a sequence of static problems. We first analyze the consumption pattern for a fixed choice of z and then analyze the optimal choice of z. The first order condition for c_A reduces to

$$(18) \quad \beta^t U'(c_A(s^t,\gamma))g(s^t) = \lambda(s^t)$$

where $\lambda(s^t)$ is the multiplier on the resource constraint. This first order condition clearly implies that all households that pay the fixed cost choose the same consumption levels so that $c_A(s^t,\gamma)$ is independent of γ. Since this problem is static this consumption level depends only on the current shock μ_t and hence we denote this consumption as $c_A(\mu_t)$.

Given that the solution only depends on current μ_t and γ we drop dependence on t. It should be clear that the optimal choice of z has a cutoff rule form: for each shock μ there is some fixed cost level $\bar{\gamma}(\mu)$ such that the households with $\gamma \leq \bar{\gamma}(\mu)$ pay this fixed cost and households with fixed costs above this level do not. For each μ, the planning problem thus reduces to choosing two numbers $c_A(\mu)$ and $\bar{\gamma}(\mu)$ to solve

$$\max U(c_A(\mu))F(\bar{\gamma}(\mu)) + U(y/\mu)(1 - F(\bar{\gamma}(\mu))$$

subject to

$$(19) \quad c_A(\mu)F(\bar{\gamma}(\mu)) + \int_0^{\bar{\gamma}} \gamma f(\gamma)d\gamma + (y/\mu)(1 - F(\bar{\gamma}(\mu)) = y.$$

The first order conditions can be summarized by

$$(20) \quad U(c_A(\mu)) - U(y/\mu) + U'(c_A(\mu))[c_A(\mu) + \bar{\gamma}(\mu) - y/\mu] = 0$$

and (19). In the appendix we show that the solution to these two equations, namely $c_A(\mu)$ and $\bar{\gamma}(\mu)$, is unique. We then have

Proposition 1: The equilibrium consumption for households is given by

$$c(s^t, \gamma) = \begin{cases} y/\mu_t & if \quad \gamma \leq \bar{\gamma}(\mu_t) \\ c_A(\mu_t) & otherwise \end{cases}$$

where the functions $\bar{\gamma}(\mu)$ and $c_A(\mu)$ are the solutions to (19) and (20).

In the decentralized economy corresponding to the planning problem asset prices are given by the multipliers on the resource constraints for the planning problem. Here from (18) these multipliers are equal to the marginal utility of active households.

Hence the pricing kernel for dollar assets is

$$(21) \quad m(s^t, s_{t+1}) = \beta \frac{U'(c_A(\mu_{t+1}))}{U'(c_A(\mu_t))} \frac{1}{\mu_{t+1}}$$

while the pricing kernel for euro assets is

$$(22) \quad m^*(s^t, s_{t+1}) = \beta \frac{U'(c_A^*(\mu_{t+1}^*))}{U'(c_A^*(\mu_t^*))} \frac{1}{\mu_{t+1}^*}.$$

These kernels can be thought of as the state contingent prices for dollars and euros normalized by the probabilities of the state.

These pricing kernels can price any dollar or euro asset. In particular, it is immediate that any asset purchased in period t with a dollar return of R_{t+1} between periods t and $t+1$ satisfies the Euler equation

$$(23) \quad 1 = E_t m_{t+1} R_{t+1}$$

where, for simplicity, here and for much of what follows we drop the s^t notation. Likewise, for every possible euro asset with rate of return R^*_{t+1} from t to $t+1$, satisfies the Euler equation

$$(24) \quad 1 = E_t m^*_{t+1} R^*_{t+1}.$$

These Euler equations immediately imply that

$$(25) \quad i_t = -\log E_t m_{t+1} \text{ and } i^*_t = -\log E_t m^*_{t+1}$$

where $\exp(i_t)$ is the dollar return on a dollar-denominated bond with interest rate i_t and $\exp(i^*_t)$ is the euro return on a euro-denominated bond with interest rate i^*_t.

The pricing kernels for dollars and euros have a natural relation, namely $m^*_{t+1} = m_{t+1}e_{t+1}/e_t$. This can be seen as follows. For every euro asset, there is a corresponding dollar asset with rate of return $R_{t+1} = R^*_{t+1}e_{t+1}/e_t$ formed when a dollar investor converts his dollars into euros at t, buys the euro asset, and converts his return back into dollars at $t+1$. Equilibrium requires that

$$E_t m_{t+1} R_{t+1} = E_t \left[m_{t+1} \frac{e_{t+1}}{e_t} \right] R^*_{t+1} = 1.$$

Since this holds for every euro return, we have that $m_{t+1}e_{t+1}/e_t$ is an equilibrium pricing kernel for euro assets. With complete markets, there can be only one euro pricing kernel, so that

$$(26) \quad \log e_{t+1} - \log e_t = \log m^*_{t+1} - \log m_{t+1}.$$

Substituting (25) and (26) into (1) it follows that

$$(27) \quad p_t = E_t \log m^*_{t+1} - E_t \log m_{t+1} - (\log E_t m^*_{t+1} - \log E_t m_{t+1}).$$

Hence, the currency risk premium depends on the difference between the "expected value of the log" and the "log of the expectation" of the pricing kernel. From Jensen's inequality, it is clear that fluctuations in the risk premium are driven by fluctuations in the conditional variability of the pricing kernel.

Finally, note that given any period 0 exchange rate e_0, (26) together with the kernels gives the entire path of the nominal exchange rate e_t. In Appendix A we show that the period 0 nominal exchange rate e_0 is given by

$$(28) \quad e_0 = \left(\bar{B} - \bar{B}_h\right) / \bar{B}^*_h.$$

Clearly, this exchange rate exists and is positive as long as $\bar{B}_h < \bar{B}$ and $\bar{B}^*_h > 0$ or $\bar{B}_h > \bar{B}$ and $\bar{B}^*_h < 0$.

4. Active Agent's Marginal Utility

It is clear from our formulas (21) and (22) for the dollar and euro pricing kernels that, in order to characterize the link between money injections and exchange rates and interest rates, we need to determine how active agents' marginal utilities in the two countries responds to money injections, namely how $U'(c_A(\mu_t))$ and $U'(c^*_A(\mu^*_t))$ vary with μ_t and μ^*_t. In the simplest monetary models (such as Lucas 1982), all of the agents in the economy are active every period and changes in money growth have no impact on marginal utilities. Our model introduces two key innovations over these simple models. First, in this model, because of the segmentation of asset markets, changes in money growth do have an impact on the consumption, and hence marginal utility, of active agents. Second, because the degree of market segmentation is endogenous, the size of this impact of changes in money growth on the marginal utility of active agents changes systematically with the level of money growth. With these two innovations, our model can deliver large and variable currency risk premia.

To study the links between money growth and asset prices in this model, we find it useful to define $\phi(\mu)$ to be the elasticity of the marginal utility of active agents to a change in money growth. With CRRA preferences of the form $U(c) = c^{1-\sigma}/(1-\sigma)$, this elasticity is given by

$$\phi(\mu) \equiv -\frac{d \log U'(c_A(\mu))}{d \log \mu} = \sigma \frac{d \log c_A(\mu)}{d \log \mu} \tag{29}$$

With $\log c_A(\mu)$ increasing in $\log \mu$, we have that $\phi(\mu) > 0$. The larger is $\phi(\mu)$, the more sensitive is the marginal utility of active agents to money growth.When $\log c_A(\mu)$ concave in $\log \mu$, $\phi(\mu)$ decreases in μ and, hence, the marginal utility of active agents is more sensitive to changes in money growth at low levels of money growth than at high levels of money growth. In this sense, the amount of risk that active agents bear depends on the level of money growth.

The following proposition shows that as money growth and inflation increase more agents become active. The result is intuitive, as inflation becomes higher, the cost of not participating is higher since the consumption of inactive agents, namely y/μ, falls as money growth μ increases.

Proposition 2. As μ increases, more agents become traders. In particular, for $\mu \geq 1$, $\bar{\gamma}'(\mu) > 0$ for $\mu > 1$, and $\bar{\gamma}'(1) = 0$.

Proof. Differentiating equations (19) and (20) with respect to μ and solving for γ' we obtain

$$\bar{\gamma}'(\mu) = \frac{\left[U'\left(\frac{y}{\mu}\right) - U'(c_A)\right]\frac{y}{\mu} - U''(c_A)(c_A + \bar{\gamma} - y/\mu)\frac{1-F}{F}y/\mu^2}{U'(c_A) - U''(c_A)(c_A + \bar{\gamma} - y/\mu) f/F}$$

where, to simplify we have omitted the arguments in the functions F, f, c_A and $\bar{\gamma}$. Notice that $c_A(1) = y$ and $\bar{\gamma}(1) = 0$. Also (20) implies that $c_A + \bar{\gamma} - y/\mu \geq 0$, with strict inequality for $\mu > 1$, and hence $U'\left(\frac{y}{\mu}\right) - U'(c_A) \geq 0$, with strict inequality for $\mu > 1$. Finally, since U is strictly concave, $U''(c_A) < 0$, then $\bar{\gamma}' > 0$ for $\mu > 1$. Using these resutls for $\mu = 1$ we get $\gamma'(1) = 0$. *Q.E.D.*

As we have noted, a key feature of our model is that consumption of active agents is increasing and concave in money growth in the relevant region of money growth rates, namely around the mean money growth rate. In the next proposition we analyze how the consumption of active agents varies with money growth around the point where there is no inflation.

Proposition 3. If $0 < F(0)$ then the log of the consumption of active agents c_A is strictly increasing and strictly concave in $\log\mu$ around $\mu = 1$. In particular,

$$\begin{aligned} \phi(1) &= \sigma \frac{d\log c_A(\mu)}{d\log\mu}|_{\mu=1} = \sigma\frac{1-F(0)}{F(0)} > 0, \\ \phi'(1) &= \sigma\frac{d^2\log c_A(\mu)}{(d\log\mu)^2}|_{\mu=1} = -\frac{\phi(1)}{F(0)} < 0. \end{aligned} \tag{30}$$

Proof. To show that $\phi(1) = \sigma(1-F(0))/F(0)$, differentiate (19) with respect to μ and $\bar{\gamma}$, and use, from the previous proposition that $\bar{\gamma}'(1) = \bar{\gamma}(1) = 0$, obtaining

$$c_A'(1) = y\frac{1-F(0)}{F(0)}.$$

Using that $c_A(1) = y$ we obtain the answer. To show that $\phi'(1) = -\phi(1)/F(0)$, first differentiate (**??**) to obtain

$$(31)\quad \phi'(1) = \sigma\left[\frac{c''_A(1)}{c_A(1)} + \frac{c'_A(1)}{c_A(1)} - \left(\frac{c'_A(1)}{c_A(1)}\right)^2\right].$$

Second, differentiate (19) with respect to μ and $\bar{\gamma}$ again, and use that at $\mu = 1$, $\bar{\gamma}'(\mu) = \bar{\gamma}(\mu) = 0$, and $c_A(\mu) + \bar{\gamma}(\mu) - y/\mu = 0$, to obtain

$$c''_A(1) = -2y\frac{1-F(0)}{F(0)}.$$

Using these expressions for c'_A and c''_A in (31) we obtain the desired result. QED

In Proposition 3 we have assumed that a fraction $F(0)$ of agents have zero cost to participate in asset markets. Notice that if the fraction of agents with zero costs is small these effects are substantial, in that the sensitivity of the marginal utility to money growth, $\phi(1)$, is large, and decreases with μ rapidily in the sense that $\phi'(1)$ is very negative.

In Propositon 2 we showed that more agents pay the fixed cost when money growth increases and in Proposition 3 we showed that locally the consumption of active agents is increasing and concave in money growth. Here we consider a simple numerical example that demonstrates these features more broadly. We let $y = 1, \sigma = 2$ and for fixed costs we let fraction $F(0) = xx$ of agents have zero fixed costs and the remainder have fixed costs with a log normal distribution $\log\gamma\tilde{}N(??,1)$. In Figures 1 and 2 we plot $F(\bar{\gamma}(\mu))$ and $\log c_A(\mu)$ against $\log\mu$ (annualized). In Figure 1 we see that as money growth increases more agents pay the fixed costs. In Figure 2 we see that the consumption of active agents is increasing and concave in money growth in the relevant range.

As we see in Figure 2, the log of consumption and hence marginal utility of active agents is a concave function of the log of money growth. Because of this non-linearity, even if we assume that our fundamental shocks, here money growth rates, have constant conditional variances, the resulting pricing kernels have time-varying conditional variances. To capture this nonlinearity in a tractable way when computing the asset prices implies by our model, we take a second order approximation to the marginal utility of active agents in (**??**)

$$\log U'\left(c_A\left(\mu_t\right)\right) = \log U'\left(c_A\left(\bar{\mu}\right)\right) - \phi\hat{\mu}_t + \frac{1}{2}\eta\hat{\mu}_t^2 \tag{32}$$

where $\hat{\mu}_t = \log \mu_t - \log \bar{\mu}$,

$$\phi \equiv -\left.\frac{d\log U'(c_A(\mu))}{d\log \mu}\right|_{\mu=\bar{\mu}} = \sigma \left.\frac{d\log c_A(\mu)}{d\log \mu}\right|_{\mu=\bar{\mu}} \tag{33}$$

$$\eta \equiv -\left.\frac{d^2\log U'(c_A(\mu))}{(d\log \mu)^2}\right|_{\mu=\bar{\mu}} = -\sigma \left.\frac{d^2\log c_A(\mu)}{(d\log \mu)^2}\right|_{\mu=\bar{\mu}}$$

Motivated by our previous results, we assume $\phi > 0$ and $\eta > 0$. With this parameterization we have the pricing kernel is given by

$$\log m_{t+1} = \log \beta/\bar{\mu} - (\phi+1)\hat{\mu}_{t+1} + \frac{1}{2}\eta\hat{\mu}_{t+1}^2 + \phi\hat{\mu}_t - \frac{1}{2}\eta\hat{\mu}_t^2. \tag{34}$$

Throughout we assume that the log of money growth in both countries follows an autoregressive process so that

$$\hat{\mu}_{t+1} = \rho\hat{\mu} + \varepsilon_{t+1} \tag{35}$$

where ε_{t+1} is independent across countries and normal with mean zero and variance σ_ε^2. Let $\sigma_\mu^2 = Var(\hat{\mu}_t - \hat{\mu}_t^*)$. A useful result for later will be that

$$Var(\hat{\mu}_t^2 - \hat{\mu}_t^{*2}) = 3\sigma_\mu^4. \tag{36}$$

5. Exchange Rates, Risk, and Interest Rates in the Model

In this section, we use our approximation (34) to examine the properties of exchange rates, currency risk premia, and interest rates in our model. We derive restrictions on the parameters of this pricing kernel required for the model to generate the main regularities observed in the data as described in section 2. We then finish the section with a numerical example in which we compute the parameters ϕ and η implied by the model and show that the model reproduces these regularities for these parameter values.

A. Exchange rates

In this section we develop restrictions on the parameters of our model so that it generates exchange rates that are roughly a martingale. We then develop a relation between money and the nominal exchange rate which will be useful for our analysis of the forward premium anomaly. In particular, we give conditions under which an increase in money growth leads to an appreciation of the home currency.

Using the result that the change in the exchange rate is related to the pricing kernel by (26), together with (34) we have that

$$E_t \log e_{t+1} - \log e_t = [\phi - (\phi+1)\rho] \left(\hat{\mu}_t^* - \hat{\mu}_t\right) + \frac{1}{2}\eta \left(\rho^2 - 1\right) \left(\hat{\mu}_t^{*2} - \hat{\mu}_t^2\right).$$

Conceptually, we find it useful to use the real exchange rate $x_t = e_t P_t^* / P_t$ to help explain our results. With symmetry between the two countries it is easy to show that the real exchange rate is given by

$$x_t = \frac{U'(c_A^*(\mu_t^*))}{U'(c_A(\mu_t))}.$$

We can then write the change in the nominal exchange rate as the sum of the change in the real exchange rate and the expected inflation differential

$$\log e_{t+1} - \log e_t = [\log x_{t+1} - \log x_t] + [\log P_{t+1}/P_t - \log P_{t+1}^*/P_t^*] \tag{37}$$

Using (21), (22) and (26) together with (16) and its foreign analog we can write the right-hand side of (37) as

$$(38) \quad [\log U'(c^*_{At+1})/U'(c_{At+1}) - \log U'(c^*_{At})/U'(c_{At})] + [\log \mu_{t+1} - \log \mu^*_{t+1}]$$

where the first bracketed term corresponds to the change in the real exchange rate and the second term corresponds to the expected inflation differential.

In the standard model with no fixed costs money growth has no effect on agents' consumption and hence the real exchange rate is not affected by changes in money growth. thus, the only effect of money growth on the exchange rate comes through the impact of money growth on inflation. As a result, in the standard model, the nominal exchange rate has the same variability as fundamentals.

In our model the real exchange will be affected by money growth. In particular, an increase in home money growth leads to increase in the consumption of the home active agents which leads the real exchange rate

$$\log x_t = \log U'(c^*_A(\mu^*_t))/U'(c_A(\mu_t))$$

to depreciate. Since the parameter ϕ measures the response of the marginal utility to money growth, for large ϕ real exchange rates will be volatile. (See Alvarez, Atkeson and Kehoe 2002 for some detailed discussion in a related model.)

Now consider the relationship between money growth and the expected changes in the nominal exchange rate. Again, in the standard model, the real exchange rate is not affected by changes in money growth, and thus, the only effect of a change in money growth on the expected change in the nominal exchange rate is the *expected inflation effect*, namely $d(E_t \log \mu_{t+1})/d \log \mu_t$. This effect is larger the more persistent is money growth. When money growth follows the autoregressive process (35) then this effect is simply ρ. Thus in the standard model an increase in money growth of one percent leads to an expected nominal depreciation of size ρ.

In our model with fixed costs an increase in money growth can lead either to an expected nominal appreciation or to an expected nominal depreciation depending on the extent of market segmentation and the persistence of money growth. To see this consider first the case where money growth is i.i.d.. In this case, the higher money growth at t has no effect on the expected money growth rate and thus has no effect on the expectation of either expected inflation $E_t \log P_{t+1}/P_t = E_t \log \mu_{t+1}$ or the expected real exchange rate

$$E_t \log x_{t+1} = E_t \log U'(c_A^*(\mu_{t+1}^*))/U'(c_A(\mu_{t+1})).$$

However, the higher money growth at t does increase the current consumption of the home active agents which leads the current real exchange rate

$$\log x_t = \log U'(c_A^*(\mu_t^*))/U'(c_A(\mu_t))$$

to appreciate. Since the expectation of the real exchange rate at $t+1$ is unchanged the exchange rate must be expected to appreciate. In this sense, the real exchange rate initially overshoots.

Consider next the case where money growth is persistent as in (35). In this case changes in the money growth rate in period t, in addition to affecting the real exchange rate in period t, also affect the predicted money growth rate at $t+1$ and thus have both expected inflation effects and effects on the expected real exchange rate in period $t+1$. Using (26),(34) and our approximation (32) we have that the impact on the expected change in the nominal exchange rate from an increase in money growth evaluated at $\mu_t = \bar{\mu}$ is

$$\frac{d}{d \log \mu_t}(E_t \log e_{t+1} - \log e_t) = -\phi(1-\rho) + \rho. \tag{39}$$

The term $-\phi(1-\rho)$ in (39) is the *expected real depreciation effect* while the term ρ is the expected inflation effect. To understand the expected real depreciation effect note that an increase in the money growth rate at t increases the consumption of home active households both on impact at t and in the subsequent period at $t+1$. The impact effect lowers the marginal utility of home goods at t by $-\phi$ and by itself makes the real exchange rate appreciate at t. This shock at t also raises the expected marginal utility of home goods at $t+1$ as well by $\rho\phi$, but if the shock is mean reverting the effect on the level of the real exchange rate at $t+1$ is smaller than it is on the real exchange rate at t so there is an expected depreciation. As money growth becomes more persistent the affects on t and $t+1$ become more similar and the amount of expected depreciation falls.

Clearly if the active agents' marginal utility decreases enough with money growth in that

$$\phi > \frac{\rho}{1-\rho} \tag{40}$$

then the expected real depreciation effect dominates the expected inflation effect and an increase in money growth leads to an expected nominal appreciation.

Finally, note that when the left-side of (40) is only slightly larger than the right-side, the expected real depreciation effect is only slightly larger than the expected inflation effect, and from (39) the nominal exchange rate is roughly a martingale. Here the expected depreciation of the real exchange rate following the initial overshooting of the real exchange rate is nearly balanced by the expected inflation effect and thus the expected change in nominal exchange rate is nearly zero.

More formally, we have that

$$Var\left(E_t \log e_{t+1} - \log e_t\right) =$$

$$\left[\phi(1-\rho)-\rho\right]^2 \sigma_\mu^2 + \frac{3}{2}\left[\left(1-\rho^2\right)\eta\sigma_\mu^2\right]^2$$

Clearly, if $\phi(1-\rho)-\rho$ is close to zero as discussed above the first term is close to zero. For the exchange rate to be roughly a martingale we also need the second term to be small. Clearly, this will be true if money growth is highly serially correlated so that ρ is close to one.

B. Risk premia

Here we discuss the impact of money growth on the risk premium. We show that the risk premium can vary systematically with the underlying shocks even though these shocks have constant conditional variances. In particular, the sensitivity of the active agents' marginal utilities to money growth shocks is smaller the higher is the level of these shocks. Intuitively, a higher level of money growth leads to lower asset market segmentation. Thus, an increase in money growth decreases the risk premium p_t. We also give conditions under which the variation in the risk premium will be large.

Recall that the risk premium can be written in terms of the pricing kernel as

$$p_t = E_t \log m_{t+1}^* - E_t \log m_{t+1} - (\log E_t m_{t+1}^* - \log E_t m_{t+1}).$$

For a conditionally lognormal variable m_{t+1}, it is well known that $\log E_t m_{t+1}$ equals $E_t \log m_{t+1} + 1/2 Var_t \log m_{t+1}$, which then implies that the risk premium p_t equals one half the difference of the conditional variances of the log kernels. Given our approximation (34) the pricing kernels are not conditionally lognormal. Nevertheless, in Appendix C we show that a similar formula applies, provided that $\eta\sigma_\varepsilon^2 < 1$. In particular, the risk premium is given by

$$p_t = \frac{1}{2}\frac{1}{1-\eta\sigma_\varepsilon^2}\left(Var_t \log m_{t+1}^* - Var_t \log m_{t+1}\right) \tag{41}$$

with

$$Var_t(\log m_{t+1}) = (\rho\eta\hat{\mu}_t - (\phi+1))^2 \sigma_\varepsilon^2 + \frac{3}{4}\eta^2\sigma_\varepsilon^4$$

and likewise for $Var_t\left(\log m_{t+1}^*\right)$

To see how the risk premium varies with money growth we calculate the derivative of the risk premium and evaluate it at $\mu_t = \bar{\mu}$ and get

$$\frac{d}{d\log\mu_t}p_t = -\frac{1}{1-\eta\sigma_\varepsilon^2}\rho\eta(\phi+1)\sigma_\varepsilon^2 < 0 \tag{42}$$

when η is positive and less than $1/\sigma_\varepsilon^2$. The basic idea here has two parts. First, since ρ is positive, a high level of money growth in period t leads agents to forecast a higher level of money growth in period $t+1$. Second, in any period, since η is positive, the marginal utility of active agents is concave in the level of money growth in that period. So as money growth increases the sensitivity of marginal utility to changes in money growth decreases. Thus, a high level of money growth in period t leads agents to predict that marginal utility in period $t+1$ will be less variable. Hence the risk premium decreases with the current level of money growth.

Next consider the unconditional variability of the risk premium. From the (41) and the expressions for $Var\,(\log m_{t+1})$ and $Var\left(\log m_{t+1}^*\right)$,

$$Var(p_t) = \left(\frac{1}{2}\frac{\eta\rho\sigma_\varepsilon^2}{1-\eta\sigma_\varepsilon^2}\right)^2 \{[2(\phi+1)]^2\,\sigma_\mu^2 + (\eta\rho)^2 3\sigma_\mu^4\}$$

Clearly, the variability of the risk premium is increasing in ϕ, η and ρ. The intuition for this result is the same as the one for (42). As these parameters increase, the conditional variance of the pricing kernel changes more with a given change in the growth rate of money.

Recall that the interest rate differential is the sum of the expected change in the exchange rate and the risk premium. If the exchange rate is a martingale, then the interest differential is equal to the risk premium and, hence, our formula for the variance of the risk premium also describes the volatility of the interest differential.

C. The Forward Premium Anomaly

To generate the forward premium anomaly we must have that the expected change in the exchange rate is positively correlated with the risk premium. In the previous two sections we have developed intuition for how an increase in money growth can lead to both an expected nominal appreciation and an decrease in the risk premium. Taken together these effects generate the first necessary condition for the forward premium anomaly namely that the risk premium is low when the currency is expected to appreciate, that is

$$cov(E_t \log e_{t+1} - \log e_t, p_t) > 0. \tag{43}$$

We have also shown under what circumstances the risk premium is variable, which is the second necessary condition. In this section we give sufficient conditions for the forward premium anomaly to hold, which as discussed in section 1 is

$$cov(E_t \log e_{t+1} - \log e_t, p_t) \geq var(E_t \log e_{t+1} - \log e_t). \tag{44}$$

We have

Proposition 4. The following conditions are sufficient for (44) to hold: i) segmentation is large enough in that ϕ satisfies

$$\phi > \frac{\rho}{1-\rho} \tag{45}$$

and ii) segmentation is sensitive enough to money growth in that η satisfies

$$\eta Var(\hat{\mu}_t) > 1. \tag{46}$$

Proof. Using

$$E_t \log e_{t+1} - \log e_t = E_t \log m^*_{t+1} - E_t \log m_{t+1}$$

and

$$p_t = E_t \log m^*_{t+1} - E_t \log m_{t+1} - (\log E_t m^*_{t+1} - \log E_t m_{t+1})$$

we can write (44) as

$$\begin{aligned} & cov(E_t \log m^*_{t+1} - E_t \log m_{t+1}, \log E_t m_{t+1} - \log E_t m^*_{t+1} - (E_t \log m^*_{t+1} - E_t \log m_{t+1}) \\ \geq \ & var(E_t \log m^*_{t+1} - E_t \log m_{t+1}). \end{aligned} \tag{47}$$

We can then use the following results to evaluate (47)

$$E_t \log m^*_{t+1} - E_t \log m_{t+1} = [\phi(1-\rho) - \rho](\hat{\mu}^*_t - \hat{\mu}_t) - \frac{1}{2}\eta\left(1-\rho^2\right)[(\hat{\mu}^*_t)^2 - (\hat{\mu}_t)^2] \tag{48}$$

which implies

$$Var\left(E_t \log m^*_{t+1} - E_t \log m_{t+1}\right) = 2\left[\phi(1-\rho) - \rho\right]^2 Var\hat{\mu}_t + \frac{\eta^2}{2}\left(1-\rho^2\right)^2 Var(\hat{\mu}^2_t). \tag{49}$$

Also, in Appendix C we show that

$$\log E_t m^*_{t+1} - \log E_t m_{t+1} = \tag{50}$$

$$E_t \log m^*_{t+1} - E_t \log m_{t+1} + \frac{1}{2}\frac{1}{1-\eta\sigma^2_\varepsilon}\left(Var_t \log m^*_{t+1} - Var_t \log m_{t+1}\right)$$

when $1-\eta\sigma^2_\varepsilon > 0$ where we can show

$$Var_t \log m^*_{t+1} - Var_t \log m_{t+1} = -2\sigma^2_\varepsilon(\phi+1)\eta\rho(\hat{\mu}^*_t - \hat{\mu}_t) + \sigma^2_\varepsilon\eta^2\rho^2[(\hat{\mu}^*_t)^2 - (\hat{\mu}_t)^2]. \tag{51}$$

From (48) we have

$$\begin{aligned} & Var\left(E_t \log m^*_{t+1} - E_t \log m_{t+1}\right) \\ = \; & [\phi(1-\rho)-\rho]^2 Var(\hat{\mu}^*_t - \hat{\mu}_t) + \frac{\eta^2}{4}\left(1-\rho^2\right)^2 Var(\hat{\mu}^{*2}_t - \hat{\mu}^2_t). \end{aligned}$$

Using (48), (51), and (50) we can write (47) as

$$\begin{aligned} & \frac{[\phi(1-\rho)-\rho]}{1-\eta\sigma^2_\varepsilon}\sigma^2_\varepsilon(\phi+1)\eta\rho Var(\hat{\mu}^*_t - \hat{\mu}_t) \\ & + \frac{1}{4}\eta\left(1-\rho^2\right)\sigma^2_\varepsilon\eta^2\rho^2\frac{1}{1-\eta\sigma^2_\varepsilon}Var\left(\hat{\mu}^{*2}_t - \hat{\mu}_t{}^2\right) \\ \geq \; & [\phi(1-\rho)-\rho]^2 Var(\hat{\mu}^*_t - \hat{\mu}_t) + \frac{\eta^2}{4}\left(1-\rho^2\right)^2 Var(\hat{\mu}^{*2}_t - \hat{\mu}^2_t) \end{aligned} \tag{52}$$

The inequality (45) ensures that $cov(E_t \log e_{t+1} - \log e_t, p_t)$ is positive. Thus, comparing separately terms on $Var(\hat{\mu}^*_t - \hat{\mu}_t)$ and $Var\left(\hat{\mu}^{*2}_t - \hat{\mu}^2_t\right)$, we have that (52) holds if

$$[\phi(1-\rho)-\rho] < \frac{\eta\sigma^2_\varepsilon}{1-\eta\sigma^2_\varepsilon}(\phi+1)\rho \tag{53}$$

and

$$\left(1-\rho^2\right) < \frac{\eta\sigma^2_\varepsilon}{1-\eta\sigma^2_\varepsilon}\rho^2. \tag{54}$$

Notice that these two equations can be written as

$$\frac{\phi(1-\rho)-\rho}{(\phi+1)\rho} < \frac{\eta\sigma^2_\varepsilon}{1-\eta\sigma^2_\varepsilon},$$

$$\frac{1-\rho^2}{\rho^2} < \frac{\eta\sigma_\varepsilon^2}{1-\eta\sigma_\varepsilon^2},$$

and that for $\phi > 0$ and $\rho \in (0,1)$,

$$\frac{\phi(1-\rho)-\rho}{(\phi+1)\rho} < \frac{1-\rho}{\rho} < \frac{1-\rho^2}{\rho^2}.$$

Thus if (54) holds, then (53) holds too.

To see that $\eta > (1-\rho^2)/\sigma_\varepsilon^2$ implies (54) observe that this inequality implies

$$\frac{1}{\rho^2} < \frac{\eta\sigma_\varepsilon^2}{(1-\rho^2)\rho^2} = \frac{\eta\sigma_\varepsilon^2}{(1-\rho^2)} + \frac{\eta\sigma_\varepsilon^2}{\rho^2}.$$

Thus

$$\frac{1-\eta\sigma_\varepsilon^2}{\rho^2} < \frac{\eta\sigma_\varepsilon^2}{(1-\rho^2)}$$

which implies (54). Q.E.D.

The sufficient conditions (45) and (46) follow naturally from our discussions in the previous sections on exchange rates and risk premiun. Recall that we found that an increase in the growth rate of the money supply leads to an expected apreciation and to a decrease in the risk premium if (40) and $\eta\sigma_\varepsilon^2\rho > 0$. Notice that (40) is identical to our first sufficient condition (45). As should be clear, the role of (45) along with $\eta\sigma_\varepsilon^2\rho > 0$ is to ensure that the covariance between expected depreciation and the risk premium in (44) is positive.

Condition (46) in the proposition ensures that the variability of the risk premium is large enough, in the sense that the covariance in (44) is larger than the variance in (44). This also follows naturally from our discussion of the variability of the risk premium in which we found that this variability is higher for larger η, ρ and σ_ε^2.

6. Some Cross-Section Implications

So far we have focused on the time-series implications of our model. Here we discuss some cross-section implications. A key mechanism at work in our model is that as the money growth rate rises so does the inflation rate and thus gains from participating in the asset market rise. As these gains rise more agents choose to be active and the amount of risk in the economy falls. In an economy with a high enough mean inflation rate, the risk in asset markets is sufficiently low that the forward premium anomaly disappears.

More precisely if the distribution of fixed costs is bounded and the risk aversion parameter $\sigma > 1$, it is easy to show there is some sufficiently inflation rate such that for all rates higher than that all agents are active and consumption is constant. In this case the model reduces to a standard one similar to that in Lucas (1982) in which risk premia are constant and the forward premium anomaly disappears.

Some evidence for this cross-section implication has been given by Bansal and Dahlquist (2000). They look at a panel of 28 emerging and developed countries and find that the forward premium anomaly is mostly present for the developed countries and mostly absent for the emerging countries. When they pool their data they find that countries with higher inflation rates tend to have smaller forward premium anomalies, in the sense that the regression coefficient b in (4) falls with the average inflation rate.

7. Conclusion

We have constructed a simple model with endogenously segmented asset markets and have shown that these frictions are a potentially important part of a complete model of exchange rates. Relative to the existing literature we make several contributions. Making market segmentation endogenous can lead to risk driving most of the movements in interest differentials, as it does in the data. Specifically, the model implies that segmentation effects, and hence risk, vary systematically with the level of inflation. In the time series this feature implies that exchange rate risk varies systematically with inflation. In the cross section this feature implies that market segmentation effects are less important in high inflation countries.

As Backus, Foresi, and Telmer (1995) and Engel (1996) have emphasized, standard monetary models have no chance of producing the forward premium anomaly because they generate a constant risk premium as long as the underlying driving processes have constant conditional variances. Backus, Foresi and Telmer argue that empirically it is unlikely that one can generate this anomaly from having nonconstant conditional variances of the driving processes. Instead they argue that what is needed is a model that generates nonconstant risk premia from driving processes that have constant conditional variances. Our model is an attempt to do exactly that.

Notes

[1]The dollar return on a euro bond, $(1+i_t^*)e_{t+1}/e_t$, is obtained by converting a dollar at t to $1/e_t$ euros, buying a euro bond paying interest $1+i_t^*$ and then converting the resulting euros back to dollars at $t+1$ at exchange rate e_{t+1}.

[2]Clearly, this reasoning only depends on the (log of the) exchange rate being approximately a martingale.

[3]This puzzle can also be stated in terms of forward exchange rates. To see this note that the forward exchange rate f_t is the price specified in a contract at t in which the buyer has the obligation to transfer at date t+1 f_t dollars and receive 1 euro. The forward premium is forward rate relative to the spot rate f_t/e_t. Arbitrage implies

$$\log f_t - \log e_t = i_t - i_t^*$$

and thus (2) can be restated as

$$Cov\left(\log f_t - \log e_t, \log e_{t+1} - \log e_t\right) < 0.$$

Thus, there is a tendency for the forward premium and the expected change in exchange rates to move in opposite directions. This observation contradicts the hypothesis that the forward rate is a good predictor of the future exchange rate.

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Appendix A

To obtain (50) we use the following result. If x is normally distributed with mean zero and variance σ^2 and satisfies $1-2b\sigma^2>0$, then

$$E\exp\left(ax+bx^2\right)=\exp\left(\frac{1}{2}\frac{a^2\sigma^2}{1-2b\sigma^2}\right)\left(\frac{1}{1-2\sigma^2 b}\right)^{1/2}.$$

To see this note that

$$E\exp\left(ax+bx^2\right)=\frac{1}{\sigma\sqrt{2\pi}}\int\exp\left(ax+bx^2\right)\exp\left(-\frac{x^2}{2\sigma^2}\right)dx=$$

$$\frac{1}{\sigma\sqrt{2\pi}}\int\exp\left(\frac{1}{2\sigma^2}\left\{2\sigma^2ax+\left(2\sigma^2b-1\right)x^2\right\}\right)dx=$$

$$\frac{1}{\sigma\sqrt{2\pi}}\int\exp\left(\frac{1}{2\sigma^2}\left\{-\left(1-2\sigma^2b\right)x^2+2\sigma^2ax-\left(\frac{\sigma^4a^2}{1-2\sigma^2b}\right)+\left(\frac{\sigma^4a^2}{1-2\sigma^2b}\right)\right\}\right)dx=$$

$$\exp\left(\frac{1}{2}\frac{a^2\sigma^2}{1-2b\sigma^2}\right)\frac{1}{\sigma\sqrt{2\pi}}\int\exp\left(-\frac{1}{2\sigma^2}\left(\left(1-2\sigma^2b\right)^{1/2}x-\frac{\sigma^2a}{(1-2\sigma^2b)^{1/2}}\right)^2\right)dx=$$

$$\exp\left(\frac{1}{2}\frac{a^2\sigma^2}{1-2b\sigma^2}\right)\frac{1}{\sigma\sqrt{2\pi}}\int\exp\left(-\frac{(1-2\sigma^2b)}{2\sigma^2}\left(x-\frac{\sigma^2a}{(1-2\sigma^2b)}\right)^2\right)dx=$$

$$\exp\left(\frac{1}{2}\frac{a^2\sigma^2}{1-2b\sigma^2}\right)\left(\frac{1}{1-2\sigma^2b}\right)^{1/2}.$$

Now, to derive (50), note that our approximation to the pricing kernel is

$$\log m_{t+1}=\log\beta-\log\bar{\mu}-(\phi+1)\hat{\mu}_{t+1}+\frac{1}{2}\eta\hat{\mu}_{t+1}^2+\phi\hat{\mu}_t-\frac{1}{2}\eta\hat{\mu}_t^2.$$

Using our assumptions that $\hat{\mu}_{t+1}=\rho\hat{\mu}_t+\varepsilon_{t+1}$ and that ε_{t+1} is normal with mean zero and variance σ_ε^2, this equation can be written

$$\log m_{t+1}=\log\beta-\log\bar{\mu}+\left(\phi(1-\rho)-\rho\right)\hat{\mu}_t-\frac{1}{2}\eta(1-\rho^2)\hat{\mu}_t^2+\left(\rho\eta\hat{\mu}_t-(\phi+1)\right)\varepsilon_{t+1}+\frac{1}{2}\eta\varepsilon_{t+1}^2$$

which implies that

$$E_t \log m_{t+1} = \log\beta - \log\bar{\mu} + (\phi(1-\rho)-\rho)\,\hat{\mu}_t - \frac{1}{2}\eta(1-\rho^2)\hat{\mu}_t^2 + \frac{1}{2}\eta\sigma_\varepsilon^2$$

and, since

$$\log m_{t+1} - E_t \log m_{t+1} = (\rho\eta\hat{\mu}_t - (\phi+1))\,\varepsilon_{t+1} + \frac{1}{2}\eta\varepsilon_{t+1}^2 - \frac{1}{2}\eta\sigma_\varepsilon^2,$$

using $E_t\varepsilon_{t+1}^4 = 3\sigma_\varepsilon^4$ and $E_t\varepsilon_{t+1}^3 = 0$, we have

$$\mathrm{Var}_t(\log m_{t+1}) = (\rho\eta\hat{\mu}_t - (\phi+1))^2\,\sigma_\varepsilon^2 + \frac{3}{4}\eta^2\sigma_\varepsilon^4.$$

Similarly,

$$\log E_t m_{t+1} = \log\beta - \log\bar{\mu} + (\phi(1-\rho)-\rho)\,\hat{\mu}_t - \frac{1}{2}\eta(1-\rho^2)\hat{\mu}_t^2 +$$

$$\log E_t \exp\left((\rho\eta\hat{\mu}_t - (\phi+1))\,\varepsilon_{t+1} + \frac{1}{2}\eta\varepsilon_{t+1}^2\right).$$

Using the result presented at the beginning of this appendix gives the last term in this expression as

$$\begin{aligned} & \log E_t \exp\left((\rho\eta\hat{\mu}_t - (\phi+1))\,\varepsilon_{t+1} + \frac{1}{2}\eta\varepsilon_{t+1}^2\right) \\ = \;& \frac{1}{2}\frac{(\rho\eta\hat{\mu}_t - (\phi+1))^2\,\sigma_\varepsilon^2}{1-\eta\sigma_\varepsilon^2} - \frac{1}{2}\log(1-\eta\sigma_\varepsilon^2). \end{aligned}$$

Equation (50) then follows directly from these equations.