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A MODEL OF CIRCULATING PRIVATE DEBT*

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ABSTRACT

We study the possible specialness of circulating as opposed to noncirculating private securities using models whose equilibria imply the existence of both. The models are pure exchange setups with spatial separation and with the potential for a variety of intertemporal trades. We find a sense in which unregulated circulating private securities are troublesome. It can happen that in order for an equilibrium to exist, the amounts of circulating debts issued at the same time in spatially and informationally separated markets have to satisfy restrictions not implied by individual maximization and market clearing in each market separately.

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One feature of economies that has been remarked upon repeatedly, mainly in discussions of "money," is the different roles played in transactions by different objects. In particular, it has been noted that some objects appear more frequently in exchange than do (most) other objects. In this paper, we present a model in which this is true for private securities; some of them circulate (get traded frequently), while others do not. Observed examples of the kinds of circulating private securities we are trying to model are bills of exchange and private bank notes.

Our ultimate goal in modeling private securities which play different roles in exchange is to address the following questions. How should we view circulating private securities? Are such securities qualitatively different from other forms of private debt, as might be suggested by a casual reading of monetary history and by proposals put forth by economists to regulate inside money or privately issued near monies? In particular, are such securities different in a way that justifies regulating their issue, while adopting laissez-faire toward other forms of private debt? Our approach to such questions is to build models or physical environments that imply the existence of private securities which play different roles in exchange and to determine what else these models or physical environments imply.

The models we study are pure exchange setups with spatial separation and possibilities for intertemporal trade. For several reasons, these are obvious models to examine. First, it has often been suggested that monies or near monies turn up when there is an absence-of-double-coincidence of wants; and absence-of-double-coincidence almost presumes spatial separation, since it implicitly refers to a situation of separate pairings among persons that have the property that no pairing can by itself produce a utility improving trade in ultimate consumption goods. Second, spatial separation is implicit or explicit in a number of recent attempts to model various aspects of monetary exchange (see Feldman (1973), Ostroy (1973), Ostroy-Starr (1974), Harris (1979), and Townsend (1980)). In this literature, distinct, spatially separated markets are used as a means of breaking up the complete-markets, general equilibrium Arrow-Debreu model, which otherwise lacks sequential trading (see Clower (1967), Hahn (1973)). Our effort differs from this literature primarily in its focus on private credit arrangements.

There are two main findings of this paper. First, and not surprisingly, we find that environments with spatially separated markets can give rise to the existence of private securities which play different roles in exchange and, in particular, can give rise to circulating private debt. Second, and, at least to us, somewhat surprisingly, we find that in some of the environments which give rise to circulating private debt, there seems to be a coordination problem with regard to the quantities of such debts issued. A coordination problem arises in part because different gross amounts of debt are consistent with the same net trades in goods. The multiplicity of equilibrium debt quantities does not present a problem in environments which imply noncirculating debts only, because in such environments the same set of agents is involved at both the issue date and the redemption date, the only times noncirculating debts are exchanged. In the case of circulating debt, however, the multiplicity can present a problem because different people trade the debts at different dates. Indeed, as we show, it can happen that in order for an equilibrium to exist, the amounts of circulating debts issued at the same time in spatially and informationally separated markets have to satisfy restrictions not implied by market clearing in each market separately. This is what we mean by a coordination problem.

The rest of the paper is organized as follows: in Section 1 we give an introductory description of the model and of two examples that we will subsequently study in some detail. In Section 2 we describe a somewhat general class of environments, formally define a competitive equilibrium for this class, and state (with the proof given in an appendix) an existence-of-equilibrium result. In Section 3, we explore the relationship between complete-markets equilibria and debt equilibria. Then, in Sections 4 and 5, respectively, we discuss the transactions-pattern implications of the theory and the coordination problem. We conclude by discussing some open questions.

1. Preliminary Description: Some Example Economies

We study set-ups with a finite number of people, each of whom lives a finite number of dates. These people meet at prescribed locations at prescribed times.

One example, which we will study closely, is an economy of four people who meet according to the pattern laid out in Table 1. We will be devoting much of our attention to the four-period version of this setup, but we will briefly mention the one, two, and three period versions of it. Note that in this example, at date 1 persons 1 and 2 are together at

Table 1: Who Meets Whom When

Date	Location 1	Location 2
1	(1,2)	(3,4)
2	(1,3)	(2,4)
3	(1,2)	(3,4)
4	(1,3)	(2,4)

location 1, while persons 3 and 4 are together at location 2. Persons 1 and 4 always stay at those locations, while persons 2 and 3 switch locations each period. Thus, for example, in the four-person, two-period economy, the economy ends after the indicated date 2 meetings, between persons 1 and 3 at location 1 and between persons 2 and 4 at location 2.

A second example we consider is an economy with three people who meet according to the pattern laid out in Table 2.

Table 2: Who Meets Whom When

Date	Location 1	Location 2
1	(1)	(2,3)
2	(3)	(1,2)
3	(2)	(1,3)
4	(1)	(2,3)
5	(3)	(1,2)

As regards commodities or consumption goods, we assume there is one commodity for each location-date combination. Equivalently, we assume that there is one good which is indexed by location and date. The set-up is pure exchange in the usual sense; goods indexed by one location-date combination cannot be transformed into goods indexed by another location-date combination; that is, there is no transportation, production, or storage technology for goods. Letting J denote the number of locations and T the number of dates, the commodity space has dimension JT. We assume that each person gets utility from commodities and has positive endowments of commodities in a proper subset whose elements correspond to the location-date combinations that the person visits. In Figures 1 and 2 we indicate by x's the subspace of the 2T commodity space that is relevant in the above sense for each of the persons in our four-person, T-period and three-person, T-period economies.

(Insert Figures 1 and 2)

As regards private securities, we let the spatial separation limit trades in securities in what we hope is a somewhat natural way. First, at a particular time, a person can only trade securities with someone he or she meets. Second, although securities can be transported, they can move only with a person. Finally, we do not allow people to renege on their debts. Securities or debts in our model take the form of promises to pay stated amounts of date and location specific goods. We assume that if the promise is presented at the relevant date and location, then it is honored.

In order to suggest how these rules and our spatial separation work, we briefly describe some of their implications for the four-person example of Table 1.

If T=2--that is, if the economy lasts only two periods--then no trade is possible in the Table 1 economy under our security trading rules. For example, person 1 cannot sell a promise to person 2 because person 2 can neither redeem it at date 2, nor pass it on to person 4, who has no use for it at date 2, the assumed last date. Note in this connection that there is a complete absence-of-double-coincidence in the Table 1 T=2 example; as shown in Figure 1, for T=2 no pair of persons has endowments and cares about a common two dimensional subspace of the commodity space. From what we have just seen, the kinds of private securities we allow do not at all overcome this particular absence-of-double-coincidence. Note, by the way, that there is potentially something to overcome in the sense that there can exist redistributions of the endowments that give rise to allocations Pareto-superior to the endowment allocation. Put differently, if all four people were together at some time "zero" and traded in complete (location and date contingent) markets, something we rule out, then the endowment would not necessarily be a competitive equilibrium.

If T=3 in the Table 1 set-up, our rules are consistent with some trade in private securities. It is easy to see, however, that only the following kinds of securities get traded: persons who meet at date 1 can trade debts due at date 3 when they meet again. For example, person 1 can issue a promise to pay location 1, date 3 good, a promise that person 2 holds until he or she again meets person 1. Thus, such securities do not circulate; they do not get traded in a secondary market and are not used to make third-party payments. Corresponding to this noncirculating characteristic is the fact that such securities do no more than accomplish trades for which there is a double-coincidence. For example, as is clear from Figure 1, persons 1 and 2 have a double-coincidence between location 1, date 1 good and location 1, date 3 good. Note also that there remains a degree of market incompleteness in the T=3 economy. In particular, the two date 2 goods cannot be traded.

If T = 4 in the Table 1 economy, then our security trading rules are consistent not only with the existence of several noncirculating securities, but also with the existence of several circulating securities. At date 1, person 1 can issue to person 2 a promise to pay location 1, date 4 good. This promise can be redeemed by being passed from person 2 to person 4 at date 2, from person 4 to person 3 at date 3, and from person 3 to person 1, the issuer, at date 4. Similarly, each of persons 2, 3, and 4 can issue at date 1 a promise of date 4 good at some location. As we will see below, except for very special cases, in this economy circulating and noncirculating debt are both necessary and sufficient for the accomplishment of all the trades achievable via complete location and date contingent markets. But this is also an economy in which a coordination problem seems to arise. As we will show in Section 5, knowledge of equilibrium prices, even present and future prices, is not enough to determine date 1 debt quantities in each market consistent with equilibrium.

2. Debt Equilibria in the General Spatial-Separation Set-Up

In this section, we describe the general class of economies under consideration. We then define a competitive debt equilibrium and prove the existence of such equilibria for the general class of economies.

We assume an economy with G persons, each of whom lives T periods. At each time t, each person g can be paired with some other person or with no one. These pairings occur at (isolated) locations. Thus, we assume that person g is assigned to some location i at each time t, and that in that location there either is or is not a single trading partner. We let there be $J \geq G/2$ locations.

If person g is in location i at time t, then he is endowed with some positive number of units, w^g_{it}, of the location i, date t consumption good. For other location-date combinations, his endowment is zero. Let w^g denote the entire JT dimensional endowment vector for person g. Also let c^g_{it} denote the nonnegative number of units of location i-date t consumption of person g and let c^g denote the entire JT dimensional consumption vector for person g. Preferences of each person g are described by a utility function $U^g(c^g)$ which is continuous, concave, and strictly increasing in the T-dimensional subspace that is relevant for g.

We restrict attention to securities that can be redeemed. Thus, if d^f_{st}, which is nonnegative, denotes securities issued by person f at time s to pay d^f_{st} units of the consumption good where f will be at time t, we consider only triplets (f,s,t) with the property that there is a path or chain of pairings leading from where f is at s to where f is at t.

We let $p^f_{st}(i,u)$ be the price per unit of d^f_{st} at location i, date u in units of good (i,u). However, we define such a price only for pairs (i,u) that potentially admit of a nontrivial trade in d^f_{st}. (This allows us to avoid having to determine a price for d^f_{st} in a market where demand and supply are identically zero and also allows us to restrict attention to positive prices.) Thus, suppose h and g meet at (i,u). We say that h is a <u>potential demander</u> of d^f_{st} at (i,u) if there is a route from h at u to f at t. We say that h is a <u>potential supplier</u> of d^f_{st} at (i,u) if there is route from f at s to h at u. We say there is a <u>market</u> in d^f_{st} at (i,u) if and only if h is a potential demander <u>and</u> g is a potential supplier at (i,u), or vice versa.

We let $d^{fg}_{st}(i,u)$ be the excess demand by g at (i,u) for d^f_{st}. In terms of this notation, our debt trading rules are

$$(1)\qquad \begin{aligned} &\sum_{u=s}^{t} d_{st}^{ff}(\cdot,u) \geq 0 \text{ for each } f \\ &\sum_{u=s}^{t'} d_{st}^{fg}(\cdot,u) \geq 0 \text{ for each } t' \geq s \text{ and } g \neq f \end{aligned}$$

where, in each case, the locations range over those that the demander visits. The first inequality says that f must end up demanding as much as f issues, which expresses our no-reneging rule. The second says that $g \neq f$ cannot supply d_{st}^{f} without having previously acquired it. Finally, as a convention, if there is not a market in d_{st}^{f} at (i,u), we set $d_{st}^{fg}(i,u) \equiv 0$.

Then, as budget constraints for any person g, we may write

$$(2)\qquad w_{iu}^{g} \geq c_{iu}^{g} + \sum d_{st}^{fg}(i,u)p_{st}^{f}(i,u)$$

there being T such constraints, one for each (i,u) that g visits. The summation in (2) is over all securites, all (f,s,t), for which a market exists at (i,u).

Now, letting d^g denote the vector of debt demands of g over <u>all</u> securities that can be issued, a vector which has many zeros, we can now give the following definition of a debt equilibrium or of a competitive, perfect-foresight equilibrium under our security trading rules.

<u>Definition</u>: A debt equilibrium is a specification of consumption and debt demands--c^g and d^g for each g = 1, 2,..., G--and positive security prices, $p_{st}^{f}(i,u)$, such that

(i) c^g and d^g maximize $U^g(c^g)$ subject to (1) and (2)

(ii) $\sum_g (c_{iu}^{g} - w_{iu}^{g}) = 0$ for each (i,u) and $\sum_g d_{st}^{fg}(i,u) = 0$ for each (i,u) and all potentially redeemable d_{st}^{f}.

In Appendix 1, we prove that every economy in our class of spatial-separation set-ups has a debt equilibrium. The proof draws heavily on Debreu (1959), although with modifications connected with the fact that the objects traded, the securities, are not ultimate consumption goods and are not bounded in an obvious way. By the way, although it may seem strange to be considering competitive (price-taking) equilibrium in markets with only two traders, everything we do also holds for set-ups in which each of our persons is a trader type and in which there are N traders of each type.

3. Debt Equilibria and Complete-Markets Equilibria

In this section we study the relationship between the equilibrium allocations and prices of complete date-location contingent markets and the allocations and prices of debt equilibria. This exercise permits us to examine whether the restrictions on trades implied by separate markets in different locations are insurmountable. In addition, the equivalence in some economies of debt equilibria and complete-markets equilibria proves useful in describing the positive implications of the theory and in revealing what we see as a coordination problem.

We first study the Table 1, T=4 economy introduced in section 1. For that economy, debt equilibria (DE's) and complete-markets equilibria (CME's) coincide in the sense that any consumption allocation which is a DE is also a CME and vice versa. We begin by showing that any CME consumption allocation can be supported by a DE.

To show that any CME can be supported by a DE in the Table 1, T = 4 economy, we start with a given CME. This we describe by individual consumption excess demands, $e^g_{it} \equiv c^g_{it} - w^g_{it}$, and by associated prices, s_{it} (in terms of an abstract unit of account). These constitute a CME if they satisfy:

(3) $\quad \sum_i \sum_t e^g_{it} s_{it} = 0$ for each g

(4) $\quad \sum_g e^g_{it} = 0$ for each (i,t)

and if, in addition, for each g, the e^g_{it}'s are utility maximizing for g subject to (3).

A corresponding DE consists of positive debt prices and nonnegative market clearing debt quantities such that (a) the debt quantities and the given CME e^g_{it}'s satisfy each person's debt budget constraints, and (b) the debt quantities and the given CME e^g_{it}'s are utility maximizing for each person given those debt prices.

Our first step is to produce candidate debt prices for the Table 1, T=4 economy. This candidate is produced by matching the terms of trade between consumption goods implied by unconstrained trades in debts to the corresponding terms of trade given by the CME prices. Thus, for example, for person 1, $p^1_{14}(1,1)$ implies a trade between location 1, date 1 good and location 1, date 4 good. (Recall that given our way of measuring debt quantities, $p^1_{14}(1,4) = 1$.) Thus, we let $p^1_{14}(1,1) = s_{14}/s_{11}$. In general, then, each debt price is taken to be a ratio of CME prices with the numerator corresponding to the redemption location-date and the denominator to the location-date of the current trade. For noncirculating debts, then, our candidate is

(5) $$(p^1_{13}(1,1),\ p^3_{13}(2,1),\ p^1_{24}(1,2),\ p^2_{24}(2,2)) =$$

$$(p^2_{13}(1,1),\ p^4_{13}(2,1),\ p^3_{24}(1,2),\ p^4_{24}(2,2))) =$$

$$(s_{13}/s_{11},\ s_{23}/s_{21},\ s_{14}/s_{12},\ s_{24}/s_{22})$$

while for circulating debts, it is

$$(6)\qquad \begin{bmatrix} p^1_{14}(1,1), & p^1_{14}(2,2), & p^1_{14}(2,3) \\ p^2_{14}(1,1), & p^2_{14}(1,2), & p^2_{14}(2,3) \\ p^3_{14}(2,1), & p^3_{14}(2,2), & p^3_{14}(1,3) \\ p^4_{14}(2,1), & p^4_{14}(1,2), & p^4_{14}(1,3) \end{bmatrix} = \begin{bmatrix} s_{14}/s_{11}, & s_{14}/s_{22}, & s_{14}/s_{23} \\ s_{24}/s_{11}, & s_{24}/s_{12}, & s_{24}/s_{23} \\ s_{14}/s_{21}, & s_{14}/s_{22}, & s_{14}/s_{13} \\ s_{24}/s_{21}, & s_{24}/s_{12}, & s_{24}/s_{13} \end{bmatrix}.$$

We can immediately indicate that this implies that satisfaction of (a) implies satisfaction of (b). To see this, multiply the debt constraint for e^g_{it} (equation (2)) by s_{it} and sum over i and t. Using (5) and (6), the result is (3), in which debt quantities do not appear. Thus, at prices given by (5) and (6), the debt constraints for any person are at least as constraining as (3). Therefore, if we can produce market clearing debt quantities, d^f_{st}'s, which make the CME e^g_{it}'s feasible choices subject to the budget constraints (2), then they are certainly utility maximizing choices. That is, (a) implies (b).

To motivate how we produce debt quantities, recall that a CME consists of arbitrary s_{it}'s and of arbitrary e^g_{it}'s that satisfy (3), (4) and zero restrictions for those e^g_{it}'s that correspond to (i,t)'s that g does not visit. For the Table 1, T=4 economy, there are 3 + 8 + 16 <u>independent</u> constraints on the 32 e^g_{it}'s. This leaves us free to choose 5 e^g_{it}'s arbitrarily, but not any 5. For example, e^1_{11} and e^2_{11} cannot both be chosen arbitrarily because (4) and the zero restrictions imply that these sum to zero. Similarly, e^1_{11}, e^1_{12}, e^1_{13}, e^1_{14} cannot each be chosen arbitrarily since (3) must be satisfied. We arrive at candidates for equilibrium debt quantities by finding some that satisfy the debt constraints for a set of e^g_{it}'s that can be chosen arbitrarily.

For the Table 1, T=4 economy, the following equations are the debt budget constraints, at prices satisfying (5) and (6), for 5 e^g_{it}'s that <u>can</u> be chosen arbitrarily:

$$(7) \qquad (e^4_{21},\ e^4_{22},\ e^1_{11},\ e^1_{12},\ e^1_{13})' = Ad$$

where

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{bmatrix} = \begin{bmatrix} s_{23}/s_{21} & 0 & 0 & 0 & 0 & 0 & -s_{14}/s_{21} & s_{24}/s_{21} \\ 0 & s_{24}/s_{22} & 0 & 0 & -s_{14}/s_{22} & 0 & s_{14}/s_{22} & 0 \\ 0 & 0 & s_{13}/s_{11} & 0 & s_{14}/s_{11} & -s_{24}/s_{11} & 0 & 0 \\ 0 & 0 & 0 & s_{14}/s_{12} & 0 & s_{24}/s_{12} & 0 & -s_{24}/s_{12} \\ 0 & 0 & -1 & 0 & 0 & 0 & -s_{14}/s_{13} & s_{24}/s_{13} \end{bmatrix}$$

and

$$d = (d^4_{13}-d^3_{13},\ d^4_{24}-d^2_{24},\ d^1_{13}-d^2_{13},\ d^1_{24}-d^3_{24},\ d^1_{14},\ d^2_{14},\ d^3_{14},\ d^4_{14})'$$

Note that zeros in the A matrix do not denote zero debt prices, but rather that the particular debt cannot be traded at the relevant location-date combination.

In order to see that there are <u>nonnegative</u> debt quantities that satisfy (7) for arbitrary s_{it}'s and an arbitrary left-hand side (LHS) of (7), it is helpful to consider an equivalent set of equations, which is obtained by replacing the last equation of (7) by itself plus a multiple (s_{11}/s_{13}) of the third equation:

$$(8) \qquad \begin{bmatrix} e^4_{21} \\ e^4_{22} \\ e^1_{11} \\ e^1_{12} \\ e^1_{13}+(s_{11}/s_{13})e^1_{11} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5+(s_{11}/s_{13})A_3 \end{bmatrix} d$$

Note that in each of the first four equations of (8), there appears (with a non-zero coefficient) a difference between (noncirculating) debts, debts which do not appear in any other equation. Thus, for any quantities of the other debts, each of the first four equations can be satisfied by choosing nonnegative quantities of the noncirculating debts which appear in that equation only. This allows us to choose nonnegative quantities of the circulating debts in any way that satisfies the last equation of (8), namely

$$(9) \qquad e^1_{13} + (s_{11}/s_{13})e^1_{11} = (s_{14}/s_{13})(d^1_{14}-d^3_{14}) - (s_{24}/s_{13})(d^2_{14}-d^4_{14})$$

Equation (9) is easily satisfied. For example, if the LHS is positive (negative), it can be satisfied by setting at zero all but d^1_{14} (d^3_{14}).

Given debt quantities that satisfy (8), all that remains is to show that they, (5) and (6), and the 11 other potentially nonzero CME e^g_{it}'s satisfy the associated debt budget constraints. Two facts imply that they do. First, if for any g, three debt budget constraints are satisfied at equality (as they are for person 1), then the fourth is also; note that we have already referred to the fact that (5) and (6) imply that the debt budget constraints satisfy (3). Second, we know that if g and h meet at (i,t), then the debt budget constraint for e^g_{it} is minus that for e^h_{it}. Thus, if debt prices and quantities are such that the debt budget constraint for g implies the CME e^g_{it}, then the debt budget constraint for h implies minus the CME e^g_{it}. But by (4), this is the CME value of e^h_{it}. This concludes our argument that any CME for the Table 1, T=4 economy can be supported by a DE.

The converse, that any DE consumption excess demands are also CME excess demands in the Table 1, T=4 economy is established in Appendix 2.

We leave to the reader the arguments establishing the equivalence between complete-markets equilibria and debt equilibria for the three person,

$T=5$ economy (see Table 2 and Figures 2 and 4). For future reference we do, however, want to describe the analogues of (7)-(9).

Note first that we can completely ignore location 1 goods in the three person economy. They do not get traded either in a debt equilibrium or in a complete-markets equilibrium. That being so, we simplify the notation by omitting location subscripts, it being understood that everything refers to location 2.

Given a CME for the three-person, $T=5$ economy, we can choose arbitrarily e_1^2, e_2^2, and e_4^2. Then, at debt prices that satisfy the analogues of (5) and (6), the corresponding debt constraints are

$$(10) \qquad (e_1^2,\ e_2^2,\ e_4^2)' = Bd$$

where

$$B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} s_4/s_1 & 0 & -s_3/s_1 & s_5/s_1 & 0 & 0 \\ 0 & s_5/s_2 & s_3/s_2 & 0 & s_4/s_2 & 0 \\ -1 & 0 & 0 & 0 & -1 & -s_5/s_4 \end{bmatrix}.$$

and

$$d = (d_{14}^2 - d_{14}^3,\ d_{25}^2 - d_{25}^1,\ d_{13}^3,\ d_{15}^2,\ d_{24}^2,\ d_{35}^1)'$$

Note that the first two elements of d involve noncirculating debts, while the other four elements are circulating debts with maturities of three or four periods. Equation (10) is equivalent to

$$(11) \qquad \begin{bmatrix} e_1^2 \\ e_2^2 \\ e_4^2 + (s_1/s_4)e_1^2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 + (s_1/s_4)B_1 \end{bmatrix} d$$

the third equation of which is

$$(12) \qquad e_4^2 + (s_1/s_4)e_1^2 = -(s_3/s_4)d_{13}^3 + (s_5/s_4)(d_{15}^2 - d_{35}^1) - d_{24}^2$$

Note that the first (second) element of d which is unconstrained in sign appears only in the first (second) equation of (11). Thus, for any choices of the last four elements of d, the first two equations of (11) can always be satisfied. It is obvious that the third equation of (11), namely (12), can be satisfied by nonnegative choices of the debts that appear in that equation.

We now want to reemphasize that DE's and CME's do not in general coincide. We shall do this by examining shorter horizon versions of the four-person and three-person economies. Since the remarks made in the introductory section take care of the four-person economy, we shall limit attention to the three person economy, starting with T=3.

In a debt equilibrium for the three-person, T=3 economy, person 2 can lend at date 1 but cannot borrow. Since a CME for that economy can have $e_1^2 > 0$, we have nonequivalence. Even for the three-person, T=4 economy there can be nonequivalence. In a debt equilibrium for that economy, negative values of e_2^2, which are possible in a CME, cannot be attained. In order that $e_2^2 < 0$, person 2 must buy a security from person 1 at date 2. But person 1 cannot issue anything that can be redeemed at date 3 or date 4 and, having been alone at date 1, has not previously acquired anything to sell.

By now the reader may have gained the impression that debt equilibria and complete-markets equilibria necessarily coincide for sufficient long-lived economies. This may well be true if pairings are sufficiently periodic. Without such a pattern, however, equivalence is not general.

4. Debt Equilibrium Transaction Patterns

We first show that some of the environments imply, rather than just permit, the existence of several private securities which play different roles in exchange. We then describe two ways of summarizing these different roles.

The coexistence of circulating and noncirculating private debts can be demonstrated using the equivalence results of section 3. That is, for some set-ups for which DE's and CME's coincide, we now show that some CME's can be supported only by DE's with both circulating and noncirculating securities.

We first demonstrate coexistence in the Table 1, T=4 set-up. From (9), if $e^1_{13}+(s_{11}/s_{13})e^1_{11} \neq 0$, then some circulating debt, d^h_{14} for some h, must be positive. Notice also that by multiplying the 5th equation of (7) by s_{13}/s_{21} and adding it to the 1st, we get an equation for $e^4_{21}+(s_{13}/s_{11})e^1_{13}$ that contains only noncirculating debts. Thus, if a Table 1, T=4 set-up is such that any CME satisfies $e^1_{13}+(s_{11}/s_{13})e^1_{11} \neq 0$ and $e^4_{21}+(s_{13}/s_{11})e^1_{13} \neq 0$, then every DE for that set-up displays positive amounts of both circulating and noncirculating debts.

Now consider the Table 2, T=5 economy. By (12), if $e^2_4+(s_1/s_4)e^2_1 \neq 0$, then some circulating debt must be positive. And, thus, by the second equation of (10), if $e^2_2 < 0$, then some noncirculating debt must be positive. Thus, if a CME satisfies both these conditions, then any DE displays positive amounts of both circulating and noncirculating debt.

In order that we not give the impression that different kinds of securities are issued only when DE's and CME's coincide, we present, by way of Tables 3 and 4, an example which displays such coexistence, but in which DE's and CME's do not coincide.

Table 3, which helps introduce the set-up for the example, describes a pattern of meetings in a six-person, T=5 economy. This set-up is equivalent to two separate three-person, T=5 set-ups of the kind already studied. Thus, DE's and CME's coincide and under the conditions given above any DE displays positive amounts of both circulating and noncirculating debts.

Table 3: Who Meets Whom When

Date	Location 1	2	3	1'	2'
1	(1)	(2,3)		(1')	(2',3')
2	(3)	(1,2)		(3')	(1',2')
3	(2)	(1,3)		(2')	(1',3')
4	(1)	(2,3)		(1')	(2',3')
5	(3)	(1,2)		(3')	(1',2')

Table 4, which is the set-up for the example, describes a pattern of meetings in another six-person, T=5 economy. This economy is meant to be identical to the Table 3 economy except for some additional pairings at location 3 and the necessary accompanying changes in endowments and preferences by location. These pairings, however, are such that the DE's for the Tables 3 and 4 economies coincide. In the Table 4 economy, persons 1 and 1' do not trade when they meet because no debt issued can be redeemed. Nor of course do persons 3 and 3' trade when they meet. But in contrast to the Table 3 economy, DE's and CME's do not in general coincide in the Table 4 set-up. In a CME for the Table 4 economy, "primes" and "unprimes" can trade with each other.

Table 4: Who Meets Whom When

Date	Location 1	2	3	1'	2'
1	(1)	(2,3)		(1')	(2',3')
2	(3)	(1,2)		(3')	(1',2')
3		(3,1)	(2,2')		(3',1')
4	(1)	(2,3)		(1')	(2',3')
5		(1,2)	(3,3')		(1',2')

We now describe two ways of summarizing the different exchange roles played by the different objects in debt equilibria in our set-ups. One way is in terms of a payments matrix (see Clower (1967)); the other is in terms of transaction velocities.

By a payments matrix we mean an N by N (symmetric) matrix, N being the number of objects we observe in a debt equilibrium; the (i,j)-th element of the matrix is "1" if object i is observed to trade for object j and is "0" otherwise. Thus, for example, for a Table 1, T=4 economy, N equals the number of distinct consumption goods, 8, plus the number of distinct private securities issued in an equilibrium. If the transaction pattern is such that each consumption good gets traded for one circulating security and one noncirculating security, then there are two nonzero elements in each row corresponding to a consumption good or to a noncirculating debt, and there are four in each row corresponding to a circulating debt. Note, by the way, that none of our nontrivial spatial set-ups gives rise to an equilibrium in which one object trades for every other object.

By the transaction velocity of an object, we mean the ratio of the average amount traded per date to the average stock, a pure number per unit time. For example, for a Table 1, T=4 economy, the following transaction velocity pattern among objects shows up in a debt equilibrium. For location i, date t consumption good, the average stock outstanding may be taken to be the total endowment divided by 4 (at dates other than t, the stock of this good is zero), while the average amount traded per date is the amount traded at t divided by 4. Thus, the transaction velocity is in the interval (0, 1). Computed in a similar way, the transaction velocity of noncirculating debt in such an economy is 2/3 (such debt is outstanding for 3 dates and the entire stock is traded at two of those dates), while that of circulating debt is unity (the maximum possible velocity given our choice of time unit).

Thus, either in terms of a payments matrix or in terms of the pattern of transaction velocities across objects, our set-ups can imply different exchange roles for different objects and, in particular, a relatively prominent exchange role for circulating private debt. Those set-ups do not, however, imply anything paradoxical or special about rate-of-return patterns across kinds of private debts. In particular, we do not find, in general, that circulating debt has a lower rate-of-return than other debt. This is easy to see in examples in which DE's and CME's coincide, for then one can express DE security prices and, hence, rates of return in terms of CME prices using relationships like (5) and (6).

5. Multiple Debt Equilibria and Coordination

As we now show, our spatial models can give rise to multiple debt equilibria and, in particular, to one kind of multiplicity which we see as a potential coordination problem. We will focus for the most part on the Table 1, T=4 economy, for which, as we showed in section 3, there is an equivalence between debt equilibria and complete-markets equilibria.

For an economy in which there is an equivalence between debt equilibria and complete-markets equilibria, there are of course at least as many debt equilibria as there are complete-markets equilibria. But that multiplicity of debt equilibria does not concern us here. Therefore, we assume that preferences and endowments are such that there is a unique complete-markets equilibrium, unique in consumptions and prices. Thus, for the Table 1, $T=4$ economy, we now assume that the set of excess demands and prices for the left-hand sides of equations (7)-(9) is unique. Even so, there are an infinite number of security trades which satisfy the right-hand sides of equations (7)-(9) and, thus, constitute debt equilibria. That is, at debt equilibrium prices which satisfy (5) and (6), there are multiple debt quantities which are maximizing for each person. Thus, there are many ways to select debt quantities for individuals in such a way that markets clear. Some aspects of this multiplicity concern us, and others do not. Our major concern is that the selection of debt quantities for individuals can require a kind of coordination or communication across markets which our security trading rules were meant to rule out <u>a priori</u>.

One way to express this coordination problem consistent with our use of a perfect foresight equilibrium concept is as follows. Let each person know the endowments and preferences of each other person. Then, in principle, each can compute equilibrium consumption excess demands and debt prices. Such knowledge of prices is consistent, however, with people in location 1 not knowing the debt quantities issued in location 2 and vice versa. To see why such knowledge is required, notice that we can write (9) in the form

$$d^1_{14} = (s_{24}/s_{14})d^2_{14} + b \tag{13}$$

where

$$(14) \qquad b = (s_{13}/s_{14})[e^1_{13} + (s_{11}/s_{13})e^1_{11}] + d^3_{14} - (s_{24}/s_{14})d^4_{14}$$

Since d^3_{14} and d^4_{14} appear in (14), a pair (d^1_{14}, d^2_{14}) that satisfies (13) cannot be determined without knowledge of at least the function of d^3_{14} and d^4_{14} that appears in (14). But nothing guides persons 3 and 4 in location 2 at t=1 to a pair d^3_{14} and d^4_{14} that gives rise to a unique value of b in (14). Indeed, any nonnegative pair (d^3_{14}, d^4_{14}) is consistent with a given net trade in consumption at t = 1 (see the first equation of (8)). Thus, if persons 1 and 2 cannot observe d^3_{14} and d^4_{14}, and persons 3 and 4 cannot observe d^1_{14} and d^2_{14}, it is hard to see how quantities that satisfy (8) can be achieved.

The coordination problem does not arise in set-ups with noncirculating debts only. In that case, the same persons are involved at both the issue date and the redemption date, so coordination between them does not seem very demanding. Formally, there <u>are</u> multiplicities in such set-ups in the sense that only differences in security trades appear in the vector of equilibrium quantities as in equation (7). But such multiplicities in effect have one person borrowing <u>and</u> lending from another at the same interest rate and seem to us to be an artifact of our formalism. Nor does the coordination problem arise in some set-ups with circulating debt, such as the three-person, T=5 economy. As is evident from equation (12), there are multiple combinations of circulating debt which would constitute a debt equilibrium. But the communication which might be required to support a particular specification does not seem inconsistent with distinct, spatial markets. That is, persons 2 and 3 might decide on d^3_{13} and d^2_{15} at date 1. Person 2 could then communicate these quantities to person 1 at date 2 before d^2_{24} is determined, and person 1 could, in turn, communicate that quantity to person 3 at date 3 before d^1_{35} is determined. Of course, in the three-person, T=5 economy there is never simultaneous issue of debt in distinct, spatial markets, as there is potentially in the Table 1, T=4 economy.

With its multiplicity of individual maximizing debt quantities and its equilibrium restriction on quantities in different markets, the Table 1, T=4 economy bears some resemblance to complete-markets models in which individual quantities are not unique, but market aggregates are. Examples are models with several assets that are perfect substitutes for individuals in equilibrium and models with increasing returns or indivisibilities. However, we think the coordination problem is different from the nonuniqueness of individual quantities in these complete-markets models. The absence of communication across markets and, hence, across subsets of the individuals, is intended to be a crucial feature of our set-ups; it plays no role in the complete-markets models.

6. Concluding Remarks

As promised, we have described a class of environments which in general gives rise to several private securities, some of which play a more prominent role in exchange than others. We have also demonstrated that in some of these environments, there is a potential problem with a laissez-faire competitive equilibrium, namely, the coordination problem described in the last section. We are, however, uncertain about whether to interpret this as a problem for actual economies or as a symptom of a defect of our assumptions.

It is tempting to take the model that gives rise to the coordination problem as an explanation of why unfettered private credit markets in actual economies can at times seem to be chaotic. Although the view that private credit markets have at times been chaotic is widely held--see, for example, Friedman's comments about private bank note issue and unfettered intermediation (1960, pages 21 and 108)--there are few, if any, successful models that describe what features of economies account for such conditions. (For two recent attempts, see Bryant (1981) and Diamond and Dybvig (1982).) Before taking the model that seriously, however, further study of it and variants of it is warranted. After all, one interpretation of the coordination problem is that we have no prediction about what would occur in a Table 1, T=4 economy.

One possible way to get such predictions is to maintain our security-trading rules while abandoning competitive behavior in favor of a Shubik-type (1973) game in which agents choose quantities taking other agents' quantity choices as given (known). Of course, we would want to impose the informational constraint that agents in one market do <u>not</u> know the quantities chosen by agents at the same time in other spatially separated markets. Another way, perhaps, is to abandon security-trading rules and to work directly with limited communication and incentive compatibility. One virtue of these approaches is that they make explicit use of the limited communication assumption. Our debt-equilibrium concept does not.

Appendix 1: Proof of Existence of Debt Equilibrium

We first prove existence with imposed arbitrary bounds on debt demands. We then argue by taking the appropriate limit that there exists some equilibrium.

Let C be the space of JT-tuples, each element of which is in $[0, w]$, where $w > w_{iu}$ for all (i,u) and where w_{iu} is the social endowment of good (i,u). Also let D be the space of n-tuples, each element of which is in $[-d,d]$, where $d > 0$ and where n is the number of elements in d^g. Note that d is the arbitrary bound on debt demands. Let P^h_{st} be the n' dimensional simplex for d^h_{st} where n' is the number of location-date combinations where there is a market for d^h_{st} and let P be the product space of the P^h_{st}'s, a finite product of finite dimensional simplexes. Finally, for each $p \in P$, let $\gamma^g(p,w^g) = \{(c^g,d^g) \in CxD$ that satisfy constraints (1) and (2)$\}$. We are now in a position to follow Debreu's (1959) proof of the existence of a standard competitive equilibrium.

Since CxD restricted by (1) is compact and convex, it follows from (1) of 4.8 of Debreu (page 63), modified for vector constraints, that $\gamma^g(p,w^g)$ is a continuous correspondence in p. The key to this assertion is that the RHS of (2) takes on the value zero at zero consumption and debt demands. Since zero is strictly less than w^g_{iu}, the exceptional case of minimum wealth does not occur.

Now, since the bounded competitive maximization problem of g involves maximizing a continuous function on a compact set, there exists a nonempty maximizing correspondence of security demands denoted $\phi^g(p)$. By theorem 4 of section 1.8 of Debreu (page 19), $\phi^g(p)$ is upper semicontinuous. It is also convex. Now let $\Phi(p) = \Sigma_g \phi^g(p)$. Clearly, $\Phi(p)$ is in the space of n-tuples, each element of which is in the interval $[-Gd,Gd]$. We denote this compact, convex set by Z.

Now consider the correspondence ρ from PxZ into itself defined by $\rho(p,z) = \mu(z)x\Phi(p)$, where $\mu(z) = \{p \in P$ which maximize $p \cdot z$ for $z \in Z\}$. Following Debreu (page 82), $\mu(z)$ is an upper semicontinuous correspondence from Z to P with $\mu(z)$ nonempty and convex. It follows that $\rho(p,z)$ is a nonempty, upper semicontinuous and convex correspondence on PxZ, which is a nonempty, compact and convex set. So ρ has a fixed point; namely, (p^*,z^*) such that $p^* \in \mu(z^*)$ and $z^* \in \Phi(p^*)$.

We now establish that $\Phi(p^*) = 0$. Consider the subvector of $\Phi(p^*)$ associated with a particular debt, d^h_{st}. If this subvector is not zero, then some element must be positive because constraint (1) does not permit the sum of these elements to be negative. So suppose some elements of the subvector are positive. The correspondence μ sets debt prices at positive levels for at least one of these positive elements and for no nonpositive element. This, in turn, implies that more of d^h_{st} is being demanded at positive prices over all location-dates than is being supplied at positive prices. This contradicts individual maximization for someone and, hence, implies that the subvector of $\Phi(p^*)$ associated with d^h_{st} is zero. It follows that $\Phi(p^*) = 0$. Moreover, a similar argument implies that no component of p^* can be zero.

We can now show that the consumptions implied by z^* and p^* satisfy market clearing for each (i,u). For each g, individual maximization implies that (2) holds with equality. So suppose we sum (2) at a given (i,u) over g. Then market clearing in consumptions follows from $\Phi(p^*) = 0$.

We have thus established the existence of a debt equilibrium with arbitrary imposed debt bounds d. Doubling these bounds, existence can again be established. Continuing in this way, one can construct a sequence of debt equilibria for economies with larger and larger debt bounds. As the associated sequence of debt equilibrium prices and consumptions has elements in the same compact space, there exists a convergent subsequence, say with limit prices and consumptions, $\bar{p}$ and $\bar{c}^g$, respectively.

Now consider the problem confronting a typical person g in the limit economy, at prices $\bar{p}$ and no imposed debt bounds whatever. The space of feasible consumptions for such a person (the budget set at prices $\bar{p}$ projected onto the space of consumptions) is compact, and the objective function is continuous, so there exists a solution, some maximizing choice of consumptions $\hat{c}^g$. Also, by construction, $\bar{c}^g$ is feasible in the limit economy. Suppose $U^g(\hat{c}^g) > U^g(\bar{c}^g)$. Then for some k^{th} economy associated with the convergent subsequence, with debt bounds sufficiently large and prices sufficiently close to $\bar{p}$, one can find a feasible consumption vector with utility arbitrarily close to $U^g(\hat{c}^g)$. But along the convergent subsequence, utility must converge to $U^g(\bar{c}^g)$. We have thus contradicted maximization for person g in the k^{th} economy. Thus $U^g(\hat{c}^g) = U(\bar{c}^g)$, and $\bar{c}^g$ solves the maximization problem of person g in the limit economy. Recall that person g was arbitrary.

We have thus established that consumptions $\bar{c}^g$ are all maximizing in the limit economy. Also, by construction, the $\bar{c}^g$ satisfy market clearing in the limit economy (this was a property of each economy in the convergent subsequence, by virtue of equilibrium). It only remains then to specify market-clearing debt demands for each person g in the budget set of person g and consistent with the choice of $\bar{c}^g$. This is done as follows. First, specify person 1's debt demands in his budget set consistent with $\bar{c}^1$. For market clearing, let these determine the debt demands of each person with whom person 1 trades, at specified dates and locations. Next, consider person 2. If there are no debt demands for him which remain to be determined, then we are done. To suppose otherwise is to contradict the fact that $\bar{c}^1$ and $\bar{c}^2$ are market clearing and the fact that person 1's debt demands are *in the budget* set of person 1. If there do remain any debts to be determined, choose these consistent with $\bar{c}^2$. Continue in this way for person 3, and so on through person G. In the end, then, we have constructed an equilibrium in the limit economy, with no bounds on debts. Recall also that the (fixed) bounds on consumptions need not be imposed in the limit economy, by the choice of the bound w.

Appendix 2: Any DE is a CME in the Table 1, T=4 Economy

Here we prove for the Table 1, T=4 economy that any DE consumption excess demands are also CME consumption excess demands.

Since the DE e_{it}^{g}'s are market clearing--that is, satisfy (4)--we have to show only that the debt budget constraints are equivalent to (3) for some choice of s_{it}'s; if we can establish that equivalence, then it follows that the DE e_{it}^{g}'s are utility maximizing subject to (3).

The debt budget constraints for person 1 in the Table 1, T=4 economy can be written

$$e_{11}^{1} = -d_{13}^{11}(1,1)p_{13}^{1}(1,1) - d_{13}^{21}(1,1)p_{13}^{2}(1,1) - d_{14}^{11}(1,1)p_{14}^{1}(1,1) - d_{14}^{21}(1,1)p_{14}^{2}(1,1)$$

$$e_{12}^{1} = -d_{24}^{11}(1,2)p_{24}^{1}(1,2) - d_{24}^{31}(1,2)p_{24}^{3}(1,2) - d_{14}^{2}(1,2)p_{14}^{2}(1,2) - d_{14}^{41}(1,2)p_{14}^{4}(1,2)$$

$$e_{13}^{1} = -d_{13}^{11}(1,3) - d_{13}^{21}(1,3) - d_{14}^{31}(1,3)p_{14}^{3}(1,3) - d_{14}^{41}(1,3)p_{14}^{4}(1,3)$$

$$e_{14}^{1} = -d_{24}^{11}(1,4) - d_{24}^{31}(1,4) - d_{14}^{11}(1,4) - d_{14}^{31}(1,4)$$

Let us add and subtract $d_{13}^{21}(1,1)p_{13}^{1}(1,1)$ on the RHS of the first equation, so that the sum $-[d_{31}^{11}(1,1) + d_{13}^{21}(1,1)]$ appears. Note that the sum $[d_{13}^{11}(1,3) + d_{13}^{21}(1,3)]$ appears in the third equation and that these sums are equal to each other because at any debt prices, debt demands satisfy $d_{13}^{11}(1,1) = -d_{13}^{11}(1,3)$ and $d_{13}^{21}(1,1) = -d_{13}^{21}(1,3)$. Moreover, the sum $[d_{13}^{11}(1,1) + d_{13}^{21}(1,1)]$ is unconstrained (as to sign). These facts imply that the first and third equations are no more constraining than the single equation that results from substituting for that sum from the 3rd equation into the first to produce

$$(A2.1) \quad e^1_{11} + p^1_{13}(1,1)e^1_{13} = d^{21}_{13}(1,1)[p^1_{13}(1,1) - p^2_{13}(1,1)] - d^{11}_{14}(1,1)p^1_{14}(1,1)$$
$$- d^{21}_{14}(1,1)p^2_{14}(1,1) - d^{31}_{14}(1,3)p^3_{14}(1,3)p^1_{13}(1,1)$$
$$- d^{41}_{14}(1,3)p^4_{14}(1,3)p^4_{14}(1,3)p^1_{13}(1,1)$$

An exactly analogous procedure allows us to combine the second and fourth equations into the following single equation which is no less constraining than those separate equations,

$$(A2.2) \quad e^1_{12} + p^1_{24}(1,2)e^1_{14} = d^{31}_{24}[p^1_{24}(1,2)-p^3_{24}(1,2)] - d^{21}_{14}(1,2)p^2_{14}(1,2)$$
$$- d^{41}_{14}\ (1,2)p^4_{14}(1,2) - p^1_{24}(1,2)[d^{11}_{14}(1,4)+d^{31}_{14}(1,4)]$$

Now consider the first term on the RHS of (A2.1). If the price difference that multiplies $d^{21}_{13}(1,1)$ is positive, then $d^{21}_{13}\ (1,1)$ is infinite. Since that cannot be an equilibrium choice, it follows that the DE prices satisfy $p^1_{13}(1,1) \quad p^2_{13}(1,1)$; that is arbitrage is not possible for person 1 in the debts d^1_{13} and d^2_{13}. And since an analogous manipulation of the debt budget constraint for person 2 implies the reverse inequality, it follows that DE prices satisfy $p^1_{13}(1,1) = p^2_{13}(1,1)$. In addition, exactly the same reasoning allows us to conclude that the DE prices satisfy the entire first equality of equation (5).

We now proceed to combine (A2.1) and (A2.2) into a single constraint that is no less constraining than both (A2.1) and (A2.2). First, add and subtract $d^{31}_{14}(1,3)p^1_{14}(1,1)$ on the RHS of (A2.1) so that the sum $d^{11}_{14}(1,1) + d^{31}_{14}(1,3)$ appears in (A2.1). This sum of demands is equal at any debt prices to $-[d^{11}_{14}(1,4) + d^{31}_{14}(1,4)]$, which appears in (A2.2). Moreover, this sum is unconstrained, implying that the equation which results from eliminating it between (A2.1) and (A2.2) is no less constraining than both

(A2.1) and (A2.2). This single equation, which we will not write out, has the following form: a linear combination of person 1's excess demands for consumption is equal to a linear combination of person 1's debt demands.

By an argument similar to that used above to establish that DE prices satisfy the first equality of equation (5)--an argument that uses the analogues of (A2.1) and (A2.2) for persons 2, 3, and 4--it follows that the DE prices must be such that the coefficient of each debt demand is zero; that is, intertemporal arbitrage among the various debts must not be possible for anyone. These restrictions on coefficients of debt demands are the ones needed in order to be able to choose s_{it}'s to satisfy equation (6). And such choices for s_{it}'s imply equivalence between the debt constraints and (3).

To summarize, we have indicated how to manipulate the debt budget constraints for the Table 1, T=4 economy so as to establish two results. The first is that security prices in a DE for that economy are constrained so that we can choose s_{it}'s to satisfy (5) and (6). Second, with that choice of s_{it}'s, debt constraints are equivalent to (3) so that the e^h_{it}'s that are utility maximizing subject to the debt constraints are also utility maximizing subject to (3). These results imply that any DE is a CME in the Table 1, T=4 economy.

Figure 1

Relevant Commodity Subspaces in the Table 1 Economy

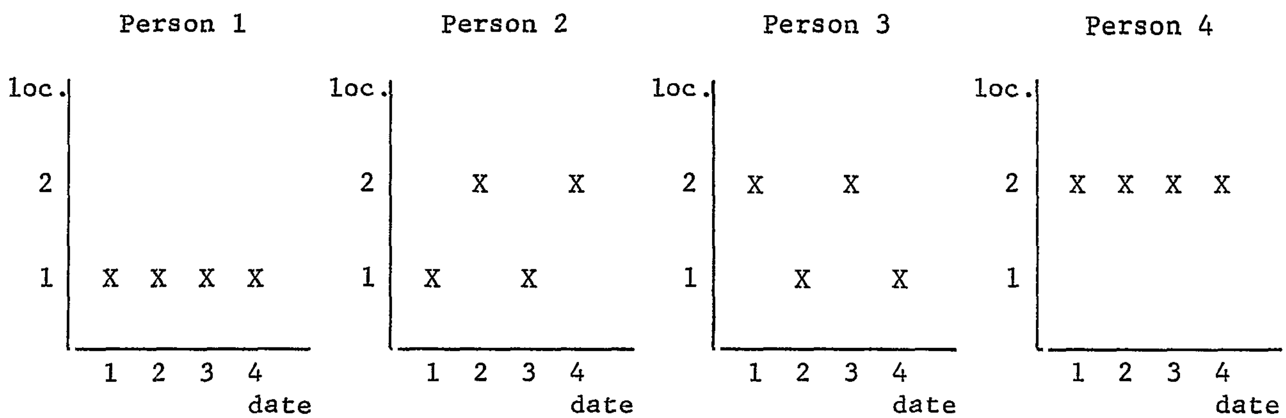
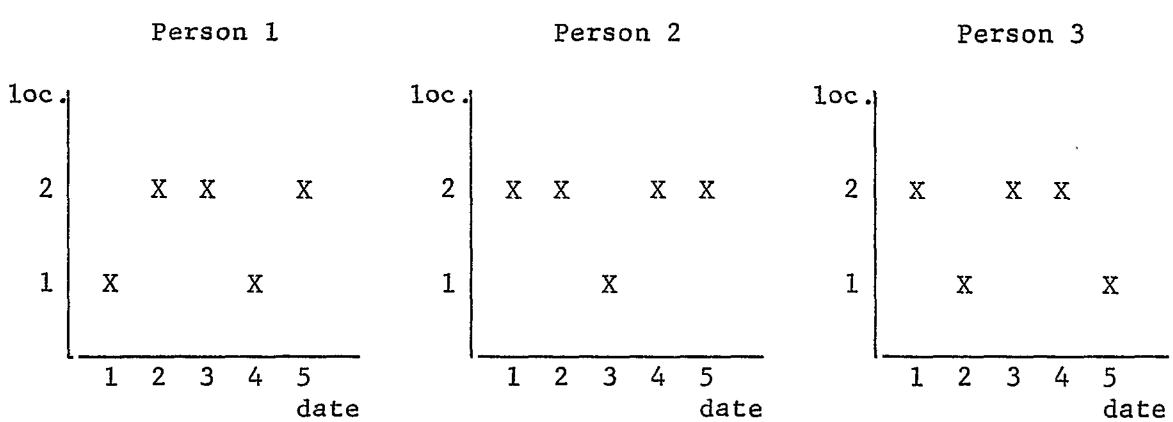


Figure 2

Relevant Commodity Subspaces in the Table 2 Economy



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